Risk Sharing, Inequality, and Fertility

Supplementary Appendix

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Contents

1 Resetting Property for a General Class of Environments

In this section, we show that the resetting property holds for a broader class of environments. Since, with private information, the resetting property at the top comes from no distortion at the top, we focus on the full information case and provide necessary and sufficient conditions for resetting to hold in a more general class of environments. Consider the model in section 2 of the paper where utility of an agent of type θ is given by $U(c_1, y, n, \theta) + \beta n^{\eta} u(c_2)$,where y is income in the first period. The specification in section 2 is a special case where $U(c_1, y, n, \theta) = u(c) + h(1 - \frac{y}{\theta} - bn)$. Moreover, suppose that having children has an additional cost $f(n, \theta)$ in terms of period 1 goods if the parent is of type θ. The planing problem, when types are public information, is given by

$$\max \sum_{i=H,L} \pi_i \left[y_i - c_i - f(n_i, \theta_i) + \frac{1}{R} n_i c_{2i} \right] \tag{1}$$

subject to

$$\begin{aligned} \sum_{i=H,L} \pi_i \left[U(c_{1i}, y_i, n_i, \theta_i) + \beta n_i^{\eta} u(c_{2i}) \right] &\geq w \\ \theta_i \geq y_i; c_{1i}, n_i, y_i, c_{2i} &\geq 0. \end{aligned}$$

The following lemma can be proved about the solution of the problem (1):

Lemma 1 *Suppose that the solution to the above problem is interior. Then*

$$\eta \frac{u(c_{2i})}{u'(c_{2i})} - c_{2i} = R f_n(n_i, \theta_i) + R \frac{U_n(c_{1i}, y_i, n_i, \theta_i)}{U_y(c_{1i}, y_i, n_i, \theta_i)}. \tag{2}$$

Proof. The first order conditions for the above problem are given by

$$\begin{aligned} \lambda U_c &= 1 \\ 1 + \lambda U_y &= 0 \Rightarrow \lambda = -\frac{1}{U_y} \\ -f_n(n_i, \theta_i) - \frac{1}{R} c_{2i} + \lambda \left[U_n + \beta \eta n_i^{\eta-1} u(c_{2i}) \right] &= 0 \\ -\frac{1}{R} n_i + \lambda \beta n_i^{\eta} u'(c_{2i}) &= 0 \Rightarrow \lambda \beta n_i^{\eta-1} = \frac{1}{R} \frac{1}{u'(c_{2i})} \end{aligned} \tag{3}$$

where λ is the multiplier on promise keeping. By replacing terms in (3), we get the following:

$$-f_n(n_i,\theta_i) - \frac{1}{R}c_{2i} - \frac{U_n}{U_y} - \frac{1}{R}\frac{u(c_{2i})}{u'(c_{2i})} = 0$$

which implies the lemma's claim. ■

Note that for section 2's specification, $f' = 0$ and $U_n/U_y = b\theta$, in which (2) becomes equation (4) in the paper. Given the above characterization for c_{2i}, one can state the following result:

Remark 2 *The resetting property, i.e., c_{2i} being independent of w, holds if and only if the function*

$$f_n(n_i,\theta_i) + \frac{U_n(c,y,n,\theta)}{U_y(c,y,n,\theta)}$$

is only a function of θ and not of (c, y, n).

The above remark implies that if we add linear goods cost of children to the model in the paper resetting still holds. Another environment that satisfies the above condition is when f is linear in n and $U = \theta u(c) + h(1 - y - bn)$. This analysis also shows that resetting property holds when there is no leisure cost of children and goods cost changes linearly with n for each type.

2 Non-Homothetic Example

In this section, we will generalize the discussion from the two period example in section 2 of the paper to non-homthetic preferences. Suppose parents have the following, non-homothetic preferences:

$$u(c_1) + h(1 - l - bn) + \beta g(n)u(c_2)$$

Assume that $g(n)u(c_2)$ is strictly increasing, strictly concave, differentiable and

$$\frac{ng'(n)/g(n)}{c_2u'(c_2)/u(c_2)} < D < \infty \quad \forall c_2, n,$$

i.e., the (negative of) elasticity of substitution between C_2 and n is uniformly bounded above. In this case, the analog of equation (4) in the paper is:

$$\frac{n(W_0,\theta_H)g'(n(W_0,\theta_H))}{g(n(W_0,\theta_H))}u(c_2(W_0,\theta_H)) = u'(c_2(W_0,\theta_H))c_2(W_0,\theta_H) + b\theta_H Ru'(c_2(W_0,\theta_H)).$$

This can be rewritten as:

$$\frac{n(W_0,\theta_H)g'(n(W_0,\theta_H))/g(n(W_0,\theta_H))}{c_2(W_0,\theta_H)u'(c_2(W_0,\theta_H))/u(c_2(W_0,\theta_H))} = 1 + \frac{b\theta_H R}{c_2(W_0,\theta_H)}.$$

Since the elasticity on the left is assumed bounded, it follows that $c_2(W_0,\theta_H)$ must be bounded away from zero for all values of W_0. It follows immediately that per capita utility – $u(c_2(W_0,\theta_H))$ – is also bounded below.

In other words, including fertility in the model will give rise to a level of per capita continuation utility that is bounded below as long as these elasticities are bounded. That is homothetic utility is not required for c_2 to be bounded away from zero. Rather what is required is that income expansion paths in (C_2, n) space should have slope that is bounded away from zero. The example given below illustrates this point.

Example. Suppose $g(n) = n^{\eta_1} + An^{\eta_2}$ with $0 > \eta_1 > \eta_2$. In this case, problem (8) in section 2 becomes the following

$$\min_{c_2,n} b\theta_H n + \frac{1}{R}nc_2$$

subject to

$$(n^{\eta_1} + An^{\eta_2})u(c_2) = W(W_0,\theta_H).$$

If we equate the marginal rate of transformation between c_2 and n to the associated marginal rate of substitution, we get

$$\frac{\eta_1 n^{\eta_1-1} + \eta_2 A n^{\eta_2-1}}{n^{\eta_1-1} + An^{\eta_2-1}}\frac{u(c_2)}{u'(c_2)} - c_2 = bR\theta_H.$$

Now suppose that W_0 and therefore, $W(W_{0,}\theta_H)$ converges to $-\infty$. In this case, one can argue that n has to converge to zero. If not, by promise keeping c_2 has to converge to 0 violating the above equation. Note that

$$\lim_{n\to 0}\frac{\eta_1 n^{\eta_1-1} + \eta_2 A n^{\eta_2-1}}{n^{\eta_1-1} + An^{\eta_2-1}} = \eta_2.$$

This means that as W_0 converges to $-\infty$, the above equation becomes:

$$\eta_2\frac{u(c_2)}{u'(c_2)} - c_2 = bR\theta_H$$

which implies that c_2 is bounded away from 0. Income exapnsion paths for this example are giveun in Figure 1. Note that at $W_0 = -\infty$, $c_2(W_0,\theta_H)$ is the slope of the income expansion path at the origin which is positive. Morevoer, $\frac{C_2}{n}$ is bounded away from zero for all points on the curve.

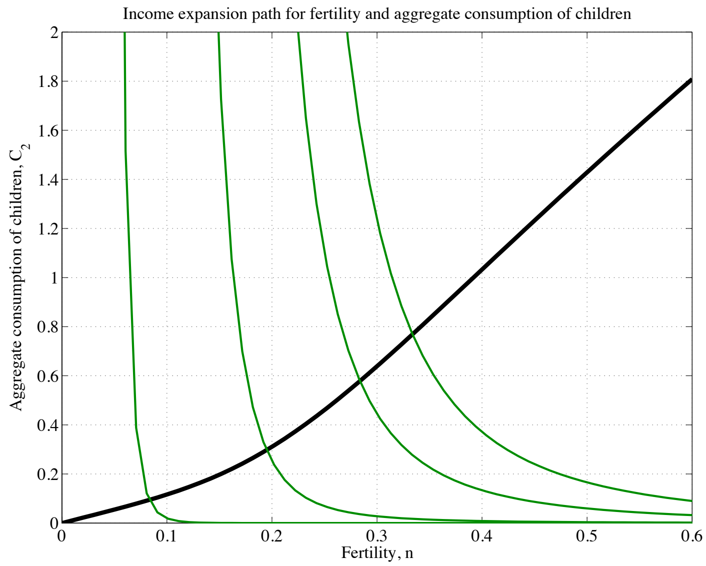

Figure 1: Income expansion path in an example with non-homothetic formulation. The slope of the income expantion path is per capita consumption. Example for $g(n) = n^{\eta_1} + An^{\eta_2}$.

3 Full Information Efficient Allocation is not Incentive Compatible

In this section, we show that the efficient allocations when there is no private information does not satisfy the incentive compatibility constraints for the maximization problem in section 3 of the paper. The efficient allocation with full information solves the following recursive problem:

$$v(w) = \min \sum_{\theta} \pi(\theta) \left[c(\theta) - \theta l(\theta) + \frac{1}{R} n(\theta) v(w'(\theta)) \right] \tag{Pfi}$$

subject to

$$\sum_{\theta} \pi(\theta)[u(c(\theta)) + h(1 - l(\theta) - bn(\theta)) + \beta n(\theta)^{\eta} w'(\theta)] \geq w.$$

This implies the following first order and Envelope conditions:

$$\begin{aligned} u'(c(\theta)) &= u'(c(\theta')) && (4)\\ \theta u'(c(\theta)) &= h'(1-l(\theta)-bn(\theta)) && (5)\\ u'(c(\theta))v'(w'(\theta)) &= \beta R n(\theta)^{\eta-1} && (6)\\ \eta v'(w'(\theta))w'(\theta) - v(w'(\theta)) &= bR\theta. && (7) \end{aligned}$$

Intuitively, from intra-family risk sharing, equation (4), we know that per capita consumption among siblings is equal. Moreover, efficiency requires that leisure is decreasing in productivity, equation (5). It is therefore sufficient to show that future utility for a low productivity agent is higher than a high productivity agent. This is shown below.

One intuition for this comes from the curvature properties of the cost function, $v(w)$. In the unconstrained efficient allocation, the planner equates per capita marginal cost $n(\theta)^{1-\eta}v'(w'(\theta))$ across various types. Notice that convexity of the above problem implies that both $v'(w)$ and $\eta v'(w)w - v(w)$ are increasing functions of w. This implies that $v'(\cdot)$ has a curvature higher than $\frac{1-\eta}{\eta}$, i.e., $\frac{v'(w)}{v'(w')} \geq \left(\frac{w}{w'}\right)^{\frac{1-\eta}{\eta}}$. Therefore, for a given relative fertility $\frac{n(\theta)}{n(\theta')} = \Delta > 1$, equating per capita marginal cost implies that relative promised utility $\frac{w'(\theta)}{w'(\theta')}$ is at most Δ^{η}. This implies that $n(\theta)^{\eta}w'(\theta) > n(\theta')^{\eta}w'(\theta')$. Hence, overall promised value, $n(\theta)^{\eta}w'(\theta)$, is higher for agents with a higher number of children (agents with lower productivity).

Hence, we can state the following lemma:

Lemma 3 *The solution to the problem (Pfi), is not incentive compatible, i.e., it violates the incentive constraint (9) in the paper.*

Proof. Consider the solution to (Pfi). From (4) and (6), we have that

$$n(\theta)^{1-\eta}v'(w'(\theta)) = n(\theta')^{1-\eta}v'(w'(\theta')), \forall \theta, \theta'. \tag{8}$$

Moreover, since $\eta v'(w)w - v(w)$ is increasing in w, we know that

$$\frac{wv''(w)}{v'(w)} > \frac{1-\eta}{\eta} \Rightarrow \frac{v''(w)}{v'(w)} > \frac{1-\eta}{\eta}\frac{1}{w}.$$

If we assume that $\theta > \theta'$, then $w'(\theta) > w'(\theta')$ and we can integrate the above equation to obtain that

$$\log\left(\frac{v'(w'(\theta))}{v'(w'(\theta'))}\right) = \int_{w'(\theta')}^{w'(\theta)} \frac{v''(w)}{v'(w)}dw > \frac{1-\eta}{\eta}\log\left(\frac{w'(\theta)}{w'(\theta')}\right)$$

and therefore

$$\frac{v'(w'(\theta))}{v'(w'(\theta'))} > \left(\frac{w'(\theta)}{w'(\theta')}\right)^{\frac{1-\eta}{\eta}}.$$

Combining the above with (8), we get the following inequality

$$\left(\frac{n(\theta')}{n(\theta)}\right)^{1-\eta} = \frac{v'(w'(\theta))}{v'(w'(\theta'))} > \left(\frac{w'(\theta)}{w'(\theta)}\right)^{\frac{1-\eta}{\eta}} \Rightarrow n(\theta')^{\eta}w'(\theta') > n(\theta)^{\eta}w'(\theta). \tag{9}$$

Moreover, since $c(\theta)$ does not depend on θ and leisure is decreasing in θ. Therefore $1-l(\theta)-bn(\theta) < 1-l(\theta')-bn(\theta') < 1-\theta' l(\theta')/\theta - bn(\theta')$, when $\theta > \theta'$. These properties together with (9) gives us the following inequality

$$\begin{aligned} &u(c(\theta)) + h(1-l(\theta)-bn(\theta)) + \beta n(\theta)^{\eta}w'(\theta) \leq \\ &\quad < u(c(\theta')) + h(1-\frac{\theta' l(\theta')}{\theta} - bn(\theta')) + \beta n(\theta')^{\eta}w'(\theta'), \forall \theta > \theta' \end{aligned}$$

which means that under the efficient allocation, agents with higher productivity would like to pretend to be low productivity. So the unconstrained efficient allocation is not incentive compatible. ■

4 Sufficiency of Downward Incentive Constraints

In this section we show, if the an allocation satisfies certain properties then downward incentive constraints are sufficient. Hence, in any solution, we can check whether the any solution of the model satisfies these conditions. These conditions are easy to check and they hold in our numerical examples that are done with two types. We provide our suffieicnt conditions in the following lemma:

Lemma 4 *Suppose an allocation $(c(\theta), l(\theta), n(\theta), w'(\theta))$ satisfies the following:*

1. *$l(\theta)\theta$ is increasing in θ ,*
2. *$1 - l(\theta) - bn(\theta) \leq 1 - \frac{\theta' l(\theta')}{\theta} - bn(\theta')$, for all $\theta > \theta'$*
3. *Local downward incentive constraints are binding:*

$$\begin{aligned} &u(c(\theta_i)) + h(1-l(\theta_i)-bn(\theta_i)) + \beta n(\theta_i)^{\eta}w'(\theta_i) = \\ &\quad u(c(\theta_{i-1})) + h(1-\frac{\theta_{i-1} l(\theta_{i-1})}{\theta_i} - bn(\theta_{i-1})) + \beta n(\theta_{i-1})^{\eta}w'(\theta_{i-1}). \end{aligned}$$

Then, incentive compatibility holds for any θ, θ'.

Proof. By part 3 of the assumption, we have

$$u(c(\theta_{i-1})) + \beta n(\theta_{i-1})^{\eta} w'(\theta_{i-1}) - u(c(\theta_i)) - \beta n(\theta_i)^{\eta} w'(\theta_i) =$$
$$= h(1 - l(\theta_i) - bn(\theta_i)) - h(1 - \frac{\theta_{i-1} l(\theta_{i-1})}{\theta_i} - bn(\theta_{i-1})).$$

By part 2 and 3 of the assumption we have

$$\frac{1}{\theta_{i-1}}(\theta_i l(\theta_i) - \theta_{i-1} l(\theta_{i-1})) \geq \frac{1}{\theta_i}(\theta_i l(\theta_i) - \theta_{i-1} l(\theta_{i-1})) \geq b(n(\theta_{i-1}) - n(\theta_i)).$$

Hence, for any $x \in [1/\theta_i, 1/\theta_{i-1}]$

$$x(\theta_i l(\theta_i) - \theta_{i-1} l(\theta_{i-1})) \geq b(n(\theta_{i-1}) - n(\theta_i))$$
$$\Rightarrow \quad 1 - x\theta_{i-1} l(\theta_{i-1}) - bn(\theta_{i-1}) \geq 1 - x\theta_i l(\theta_{i-1}) - bn(\theta_i).$$

Therefore, using part 1 and concavity of $h(\cdot)$,

$$-h'(1 - x\theta_{i-1} l(\theta_{i-1}) - bn(\theta_{i-1}))\theta_{i-1} l(\theta_{i-1}) \geq -h'(1 - x\theta_i l(\theta_{i-1}) - bn(\theta_i))\theta_i l(\theta_i).$$

Integrating both sides from $1/\theta_i$ to $1/\theta_{i-1}$, we get

$$h(1-l(\theta_{i-1})-bn(\theta_{i-1}))-h(1-\frac{\theta_{i-1} l(\theta_{i-1})}{\theta_i}-bn(\theta_{i-1})) \geq h(1-\frac{\theta_i l(\theta_i)}{\theta_{i-1}}-bn(\theta_i))-h(1-l(\theta_i)-bn(\theta_i)).$$

Therefore,

$$u(c(\theta_{i-1})) + \beta n(\theta_{i-1})^{\eta} w'(\theta_{i-1}) - u(c(\theta_i)) - \beta n(\theta_i)^{\eta} w'(\theta_i) \geq$$
$$\geq h(1 - \frac{\theta_i l(\theta_i)}{\theta_{i-1}} - bn(\theta_i)) - h(1 - l(\theta_{i-1}) - bn(\theta_{i-1})).$$

Hence, the local upward incentive constraints are satisfied.

Now, we will show that other upward incentive constraints are satisfied . To illustrate we show the argument for i and $i+2$ and a similar inductive argument works for higher differences. By condition 2, we know that:

$$\frac{1}{\theta_{i+2}}(\theta_{i+2} l(\theta_{i+2}) - \theta_{i+1} l(\theta_{i+1})) \geq b(n(\theta_{i+1}) - n(\theta_{i+2}))$$

and therefore,

$$\frac{1}{\theta_i}(\theta_{i+2}l(\theta_{i+2})-\theta_{i+1}l(\theta_{i+1}))\geq\frac{1}{\theta_{i+1}}(\theta_{i+2}l(\theta_{i+2})-\theta_{i+1}l(\theta_{i+1}))\geq b(n(\theta_{i+1})-n(\theta_{i+2})).$$

Hence, for any $x\in[1/\theta_{i+1},1/\theta_i]$,

$$1-x\theta_{i+1}l(\theta_{i+1})-bn(\theta_{i+1})\geq 1-x\theta_{i+2}l(\theta_{i+2})-bn(\theta_{i+2})$$

and we have,

$$\begin{aligned}h'(1-x\theta_{i+1}l(\theta_{i+1})-bn(\theta_{i+1})) &\leq h'(1-x\theta_{i+2}l(\theta_{i+2})-bn(\theta_{i+2}))\\ \Rightarrow -h'(1-x\theta_{i+1}l(\theta_{i+1})-bn(\theta_{i+1}))\theta_{i+1}l(\theta_{i+1}) &\geq -h'(1-x\theta_{i+2}l(\theta_{i+2})-bn(\theta_{i+2}))\theta_{i+2}l(\theta_{i+2}).\end{aligned}$$

So,

$$\begin{aligned}&h(1-\frac{\theta_{i+1}l(\theta_{i+1})}{\theta_i}-bn(\theta_{i+1}))-h(1-l(\theta_{i+1})-bn(\theta_{i+1})) \\ &\geq h(1-\frac{\theta_{i+2}l(\theta_{i+2})}{\theta_i}-bn(\theta_{i+2}))-h(1-\frac{\theta_{i+2}l(\theta_{i+2})}{\theta_{i+1}}-bn(\theta_{i+2})).\end{aligned}\tag{10}$$

Rewriting local IC's for $i,i+1$ and $i+1,i+2$:

$$\begin{aligned}&u(c(\theta_i))+\beta n(\theta_i)^\eta w'(\theta_i)-u(c(\theta_{i+1}))-\beta n(\theta_{i+1})^\eta w'(\theta_{i+1})\geq \\ &\geq h(1-\frac{\theta_{i+1}l(\theta_{i+1})}{\theta_i}-bn(\theta_{i+1}))-h(1-l(\theta_i)-bn(\theta_i))\end{aligned}\tag{11}$$

$$\begin{aligned}&u(c(\theta_{i-1}))+\beta n(\theta_{i+1})^\eta w'(\theta_{i+1})-u(c(\theta_{i+2}))-\beta n(\theta_{i+2})^\eta w'(\theta_{i+2})\geq \\ &\geq h(1-\frac{\theta_{i+2}l(\theta_{i+2})}{\theta_{i+1}}-bn(\theta_{i+2}))-h(1-l(\theta_{i+1})-bn(\theta_{i+1})).\end{aligned}\tag{12}$$

Summing over inequalities (10)-(12), we get:

$$\begin{aligned}&u(c(\theta_{i-1}))+\beta n(\theta_i)^\eta w'(\theta_i)-u(c(\theta_{i+2}))-\beta n(\theta_{i+2})^\eta w'(\theta_{i+2})\geq \\ &\geq h(1-\frac{\theta_{i+2}l(\theta_{i+2})}{\theta_i}-bn(\theta_{i+2}))-h(1-l(\theta_i)-bn(\theta_i))\end{aligned}$$

which is the upward incentive constraint for $i,i+2$. The rest of the upward and downward incentive constraints can be proved in a similar way. ■

The conditions provided are very intuitive. The first condition asserts that income has to be increasing in type. A similar condition arises in most Mirrleesian environments. The second assumption implies that leisure from lying downward is higher than leisure under telling the truth. Notice that, in environments without fertility, conditions 1 and 2 are equivalent. However, since fertility is endogenous and potentially different for different types condition 2 is needed. Condition 3 is very common in the literature can sometimes shown to be binding. This lemma is similar to a result in mechanism design that provides sufficiency of local incentive constraints, see Matthews and Moore (1987), and Pavan et al. (2009).

5 Spreading of Future Promises

In this section, we show that when $\beta R = 1$, then it is optimal for the planner to spread continuation utility for the highest and lowest values of shock. This is in fact very similar to lemma 5 in Thomas and Worrall (1990). Therefore, it suggests that the same proof as theirs would work to show that total continuation utility, $N_t^{1-\eta} w_t$ converges to its lowest bound, when $\beta R = 1$.

Lemma 5 *Suppose that $\beta R = 1$. Then $n(w,\theta_1)^{1-\eta} v'(w'(w,\theta_1)) \leq v'(w) \leq n(w,\theta_I)^{1-\eta} v'(w'(w,\theta_I))$, for all w. Moreover, if $\mu(I,j)$ $(\mu(j,1))$ is positive for some j, $v'(w) < n(w,\theta_I)^{1-\eta} v'(w'(w,\theta_I))$ $(v'(w) > n(w,\theta_1)^{1-\eta} v'(w'(w,\theta_1)))$.*

Proof. The FOC w.r.t $w'(\theta_1)$ and $w'(\theta_I)$ are given by:

$$\begin{aligned} \beta n(\theta_1)^{\eta-1}\left[\lambda\pi(\theta_1) - \sum_{i=2}^{I}\mu(i,1)\right] &= \frac{1}{R}\pi(\theta_1)v'(w'(\theta_1)) \\ \beta n(\theta_I)^{\eta-1}\left[\lambda\pi(\theta_I) + \sum_{i=1}^{I-1}\mu(I,i)\right] &= \frac{1}{R}\pi(\theta_I)v'(w'(\theta_I)) \end{aligned}$$

where $\mu(i,j) \geq 0$ is the Lagrange multiplier on the incentive constraint i,j with $i > j$ and $\lambda > 0$ is the multiplier associated with the promise-keeping constraint. Since $\beta R = 1$, the above equalities imply that $n(\theta_1)^{1-\eta} v'(w'(\theta_1)) \leq \lambda \leq n(\theta_I)^{1-\eta} v'(w'(\theta_I))$ with the inequalities being strict if and only if one of the multipliers $\mu(I,i)$ and $\mu(i,1)$ are positive. By the Envelope theorem, we know that $\lambda = v'(w)$. This proves the claim. ■

6 Existence of Bounded Ergodic Set

Assumption 6 *For any given w, if $l(w,\theta_i) = 0$ for some $1 \leq i \leq I$, then $l(w,\theta_j) = 0$ for all $j < i$.*

Assumption 7 *The policy function $w'(w,\theta)$ is continuous for all θ.*

Proposition 8 *Suppose $V(N, W)$ is convex and continuously differentiable and assumptions 6 and 7 hold. The, there exist $\bar{w} < 0$ and $\underline{w} > -\infty$ such that for every $w \in [\underline{w}, \bar{w}]$, $w'(w, \theta)$ belongs to $[\underline{w}, \bar{w}]$ for all $\theta \in \Theta$.*

Suppose there are I types $\Theta = \{\theta_1, \ldots, \theta_I\}$ and $\theta_{i+1} > \theta_i$ for all $1 < i \leq I$. We break the proof into few lemmas.

Consider the following problem

$$
\begin{aligned}
v(w) = \min_{c(\theta), l(\theta), n(\theta)} \quad & \sum_{\theta \in \Theta} \pi(\theta) \left(c(\theta) - \theta l(\theta) + \frac{1}{R} n(\theta) v(w'(\theta)) \right) \\
\text{s.t.} \quad & \sum_{\theta \in \Theta} \pi(\theta) \left(u(c(\theta)) + h(1 - l(\theta) - bn(\theta)) + \beta n(\theta)^{\eta} w'(\theta) \right) \geq w \\
& u(c(\theta_i)) + h(1 - l(\theta_i) - bn(\theta_i)) + \beta n(\theta_i)^{\eta} w'(\theta_i) \geq \\
& u(c(\theta_j)) + h\left(1 - \frac{\theta_j l(\theta_j)}{\theta_i} - bn(\theta_j)\right) + \beta n(\theta_j)^{\eta} w'(\theta_j); \quad \forall i, \forall j < i.
\end{aligned}
$$

We have shown in Lemma 1 in the paper that convexity and differentiability of $V(N, W)$ implies that $v(w)$ is differentiable and convex. It also implies that $\eta w v'(w) - v(w)$ is an increasing function of w.

Lemma 9 *For any w such that $l(w, \theta_I) > 0$ we have $w'(w, \theta_i) < w'(w, \theta_I)$ for all $i < I$.*

Proof. Let λ and $\mu(i, j)$ be multipliers on promise keeping and incentive constraint of type θ_i who wants to pretend to be of type θ_j. For now suppose $l(w, \theta_i) > 0$ for all i. First order conditions are (we suppress the dependence of the allocation on w, it plays no role in the following arguments):

$$
\theta_I \pi(\theta_I) = \left(\lambda \pi(\theta_I) + \sum_{j=1}^{I-1} \mu(I, j) \right) h'(1 - l(\theta_I) - bn(\theta_I)) \tag{13}
$$

$$
\begin{aligned}
\theta_i \pi(\theta_i) = & \left(\lambda \pi(\theta_i) + \sum_{j=1}^{i-1} \mu(i, j) \right) h'(1 - l(\theta_i) - bn(\theta_i)) \\
& - \sum_{k=i+1}^{I} \mu(k, i) \frac{\theta_i}{\theta_k} h' \left(1 - \frac{\theta_i l(\theta_i)}{\theta_k} - bn(\theta_i) \right)
\end{aligned} \tag{14}
$$

$$\pi(\theta_I)v'(w'(\theta_I)) = \left(\lambda\pi(\theta_I) + \sum_{j=1}^{I-1}\mu(I,j)\right)\beta R n(\theta_I)^{\eta-1} \tag{15}$$

$$\pi(\theta_I)v'(w'(\theta_i)) = \left(\lambda\pi(\theta_i) + \sum_{j=1}^{i-1}\mu(i,j) - \sum_{k=i+1}^{I}\mu(k,i)\right)\beta R n(\theta_i)^{\eta-1} \tag{16}$$

$$\begin{aligned}\pi(\theta_I)v(w'(\theta_I)) &= \left(\pi(\theta_I)\lambda + \sum_{j=1}^{I-1}\mu(I,j)\right)\left(\eta\beta R n(\theta_I)^{\eta-1}w'(\theta_I) - Rbh'(1-l(\theta_I)-bn(\theta_I))\right)\\ \pi(\theta_i)v(w'(\theta_i)) &= \left(\lambda\pi(\theta_i) + \sum_{j=1}^{i-1}\mu(i,j)\right)\left(\eta\beta R n(\theta_i)^{\eta-1}w'(\theta_i) - Rbh'(1-l(\theta_i)-bn(\theta_i))\right)\\ &\quad - \sum_{k=i+1}^{I}\mu(k,i)\left(\eta\beta R n(\theta_i)^{\eta-1}w'(\theta_i) - Rbh'\left(1-\frac{\theta_i l(\theta_i)}{\theta_k} - bn(\theta_i)\right)\right) \end{aligned} \tag{17}$$

for $1 \le i < I$.

Combining these equations we can get the following

$$\eta w'(\theta_I)v'(w'(\theta_I)) - v(w'(\theta_I)) = Rb\theta_I \tag{18}$$

$$\begin{aligned}\eta w'(\theta_i)v'(w'(\theta_i)) - v(w'(\theta_i)) &= Rb\left(\lambda + \frac{1}{\pi(\theta_i)}\sum_{j=1}^{i-1}\mu(i,j)\right)h'(1-l(\theta_i)-bn(\theta_i))\\ &\quad -\frac{Rb}{\pi(\theta_i)}\sum_{k=i+1}^{I}\mu(k,i)h'\left(1-\frac{\theta_i l(\theta_i)}{\theta_k} - bn(\theta_i)\right). \end{aligned} \tag{19}$$

Since $\eta w v'(w) - v(w)$ is increasing in w, to establish the claim of the lemma it is enough to show that the right hand side of the equation (19) is smaller that $Rb\theta_I$. But notice that

$$\begin{aligned} &Rb\left(\lambda + \frac{1}{\pi(\theta_i)}\sum_{j=1}^{i-1}\mu(i,j)\right)h'(1-l(\theta_i)-bn(\theta_i)) -\\ &\frac{Rb}{\pi(\theta_i)}\sum_{k=i+1}^{I}\mu(k,i)h'\left(1-\frac{\theta_i l(\theta_i)}{\theta_k} - bn(\theta_i)\right) <\\ &Rb\left(\lambda + \frac{1}{\pi(\theta_i)}\sum_{j=1}^{i-1}\mu(i,j)\right)h'(1-l(\theta_i)-bn(\theta_i)) -\\ &\frac{Rb}{\pi(\theta_i)}\sum_{k=i+1}^{I}\frac{\theta_i}{\theta_k}\mu(k,i)h'\left(1-\frac{\theta_i l(\theta_i)}{\theta_k} - bn(\theta_i)\right) = Rb\theta_i < Rb\theta_I. \end{aligned}$$

The first inequality follows from the fact that $\frac{\theta_i}{\theta_k} < 1$ for $k > i$. The rest follows from (14). This finishes the proof for the case in which $l(w, \theta_i) > 0$.

Now consider the case in which the non-negativity constraint on hours is binding for some types $1 \leq i < I$. Then by assumption 6 for all types θ_j ,$1 \leq j < i$ we have $l(w, \theta_j) = 0$. Then all types θ_i, $1 \leq i < j$ receive the same allocations and therefore $\mu(i, j) = 0$ for $1 \leq j < i$. The equations (14) and (19) for type θ_i change to

$$\theta_i < \left(\lambda - \frac{1}{\pi(\theta_i)} \sum_{k=i+1}^{I} \mu(k,i) \frac{\theta_i}{\theta_k}\right) h'(1 - bn(\theta_i))$$

and

$$\eta w'(\theta_i) v'(w'(\theta_i)) - v(w'(\theta_i)) = Rb \left(\lambda - \frac{1}{\pi(\theta_i)} \sum_{k=i+1}^{I} \mu(k,i)\right) h'(1 - bn(\theta_i))$$

Suppose $w'(\theta_i) > w'(\theta_I)$, then

$$Rb\theta_I < \eta w'(\theta_i) v'(w'(\theta_i)) - v(w'(\theta_i)) = Rb \left(\lambda - \frac{1}{\pi(\theta_i)} \sum_{k=i+1}^{I} \mu(k,i)\right) h'(1 - bn(\theta_i))$$

and therefore

$$\begin{aligned} h'(1 - bn(\theta_i)) &> \frac{\theta_I}{\lambda - \frac{1}{\pi(\theta_i)} \sum_{k=i+1}^{I} \mu(k,i)} \\ &> \frac{\theta_I}{\lambda + \frac{1}{\pi(\theta_I)} \sum_{j=1}^{I-1} \mu(I,j)} = h'(1 - bn(\theta_I) - l(\theta_I)). \end{aligned}$$

Hence

$$1 - bn(\theta_i) < 1 - l(\theta_I) - bn(\theta_I).$$

On the other hand $w'(\theta_i) > w'(\theta_I)$ implies

$$\left(\lambda - \frac{1}{\pi(\theta_i)} \sum_{k=i+1}^{I} \mu(k,i)\right) n(\theta_i)^{\eta-1} = \frac{v'(w'(\theta_i))}{\beta R} > \frac{v'(w'(\theta_I))}{\beta R} = \left(\lambda + \frac{1}{\pi(\theta_I)} \sum_{j=1}^{I-1} \mu(I,j)\right) n(\theta_I)^{\eta-1}$$

and therefore

$$n(\theta_I) > n(\theta_i).$$

These implies that $l(\theta_I)$ has to be negative which is a contradiction. Therefore, we must have $w'(w, \theta_i) < w'(w, \theta_I)$ for all w such that $l(w, \theta_I) > 0$. By assumption 6 $l(w, \theta_j) = 0$ for all $j < i$, and we know that $w'(w, \theta_j) = w'(w, \theta_i) < w'(w, \theta_I)$ for all $j < i$ (since all types θ_j receive the same allocations for $1 < j \leq i$). ■

Next we will find the promised utility at which the non-negativity for type θ_I just binds. At this point, all types work zero hours and therefore receive the same allocation (assumption 6).

Let $\hat{c}$, $\hat{n}$ and w_I solve the following equations

$$\begin{aligned} \theta_I u'(\hat{c}) &= h'(1 - b\hat{n}) \\ v'(w_I) u'(\hat{c}) &= \beta R \hat{n}^{\eta - 1} \end{aligned}$$

$$\eta w_I v'(w_I) - v(w_I) = Rb\theta_I.$$

Define

$$\hat{w} = u(\hat{c}) + h(1 - b\hat{n}) + \beta \hat{n}^\eta w_I.$$

Note that $w'(\hat{w}, \theta_I) = w'(\hat{w}, \theta_i) = w_I$ for all i.

Our goal is to show that for $w > \hat{w}$, it is optimal for type θ_I to work zero hours for all θ. After we establish that, we can prove the claim of the proposition for two cases on $w_I > \hat{w}$ and $w_I < \hat{w}$.

Lemma 10 *If $w > \hat{w}$, then $l(w, \theta_I) = 0$.*

Proof. Suppose otherwise and consider the following equations

$$w = u(c) + h(m) + \beta n^\eta w_I$$

and

$$\begin{aligned} \theta_H u'(c) &= h'(m) \\ v'(w_I) u'(c) &= \beta R n^{\eta - 1} \end{aligned}$$

in which $m = 1 - l - bn$. Also, the first order conditions at $\hat{w}$ are

$$\begin{aligned} \theta_I u'(\hat{c}) &= h'(\hat{m}) \\ v'(w_I) u'(\hat{c}) &= \beta R \hat{n}^{\eta - 1} \end{aligned}$$

$$\hat{w} = u(\hat{c}) + h(1 - b\hat{n}) + \beta \hat{n}^{\eta} w_I$$

in which $\hat{m} = 1 - b\hat{n}$. Subtract the first order conditions at w and $\hat{w}$ to obtain:

$$w - \hat{w} = u(c) - u(\hat{c}) + h(m) - h(\hat{m}) + \beta w_I(\hat{n}^{\eta} - n^{\eta}) \tag{20}$$

$$\begin{aligned} \theta_H(u'(c) - u'(\hat{c})) &= h'(m) - h'(\hat{m}) \\ v'(w_I)(u'(c) - u'(\hat{c})) &= \beta R(n^{\eta-1} - \hat{n}^{\eta-1}). \end{aligned}$$

Then, by concavity of $u(c)$, $h(m)$ and n^{η} we know that $(c-\hat{c})$,$(m-\hat{m})$ and $(n-\hat{n})$ must all have the same signs.

Also, from (20) we have

$$u'(\hat{c})(c - \hat{c}) + h(m) + h'(\hat{m})(m - \hat{m}) + \beta w_I \eta \hat{n}^{\eta-1}(n - \hat{n}) > w - \hat{w} > 0.$$

This implies that $(c-\hat{c})$,$(m-\hat{m})$ and $(n-\hat{n})$ must be all positive. If $m > \hat{m}$ and $n > \hat{n}$, then we must have $l = 1 - m - bn < 1 - \hat{m} - b\hat{n} = 0$. This is a contradiction, therefore, the non-negativity constraint on hours must be binding at $\hat{w}$. ■

Lemma 11 *If $w_I > \hat{w}$, there exist $\hat{w} \leq w^* \leq 0$ such that $w'(w^*, \theta) = w^*$.*

Proof. Recall that that since $l(w, \theta) \geq 0$ is binding, by assumption 6 all types work zero hours and receive the same allocations. Therefore, the incentive constraint is slack. The first order conditions for type θ_I are

$$\lambda h'(1 - bn(\theta_I)) > \theta_I$$

and

$$v(w'(\theta_I)) + Rb\lambda h'(1 - bn(\theta_I)) = \lambda\eta\beta Rn(\theta_I)^{\eta-1}w'(\theta_I).$$

Therefore

$$\eta w'(\theta_I)v'(w'(\theta_I)) - v(w'(\theta_I)) = Rb\lambda h'(1 - bn(\theta_I)) > Rb\theta_I.$$

This implies $w'(w, \theta_I) > w_I > \hat{w}$. Define the function $w'_{\epsilon}(\cdot, \theta) : [\hat{w}, -\epsilon] \to [\hat{w}, -\epsilon]$ as

$$w'_{\epsilon}(\cdot, \theta) = \begin{cases} w'(\cdot, \theta) & \text{if } w'_{\epsilon}(\cdot, \theta) \leq -\epsilon \\ -\epsilon & \text{if } w'_{\epsilon}(\cdot, \theta) > -\epsilon \end{cases}.$$

This function must have a fixed point $w^*_\epsilon \in [\hat{w}, -\epsilon]$. We know that $w'(\cdot,\theta) = \lim_{\epsilon\to 0} w'_\epsilon(\cdot,\theta)$. Then, either a $\hat{w} < w^* < 0$ exists such that $w'(w^*,\theta) = w^*$ or $\lim_{w\to 0} w'(w,\theta) = 0$. (Note that because no one works all types receive the same allocations). ■

So far we have established that if $w_I > \hat{w}$, then we can choose $\underline{w} = \bar{w} = w^*$ and the proposition is proved.

Now suppose $w_I \leq \hat{w}$. By lemma 9, we have $w(w,\theta) \leq w_I \leq \hat{w}$ for all w. Let $\bar{w} = \hat{w}$ and notice that proposition 3 together with continuity of $w'(w,\theta_i)$ (assumption 7) implies that there exist a $\tilde{w}$ small enough so that $w'(\tilde{w},\theta_i) > -\infty$. By assumption 7, $w'(w,\theta_i)$ is continuous in w and hence has a minimum $\underline{w}$ in $[\tilde{w},\bar{w}]$. This concludes the proof of the proposition.

7 A Uniqueness Result

In this example, we assume that resetting property at the top holds for every $w < w_0$. For this case, using a direct constructive proof, we show that the model implies a long-run distribution for per capita consumption and characterize its properties. What is convenient about this example is that we can show that if resetting holds at all w and Assumption 8 holds with $\overline{w} = w_0$, then there is a unique stationary distribution within a certain class.

We know that $w_0 = w'(w,\theta_H)$ is independent of w for all $w \leq w_0$. Hence, we can define the following set of promised values:

$$\mathcal{W} = \{w_n | w_{n+1} = w'(w_n,\theta_L), \forall n \geq 0\}$$

By Corollary 7 in the paper, there is a lower bound $\underline{w}$ such that $\underline{w} \leq w \leq w_0$, for all $w \in W$.

Assumption 12 *Assume that $w_j \neq w_i$ if $j \neq i$.*

Consider a distribution over W, $\Psi = (\psi_0,\psi_1,\cdots)$ with $\sum_{i=0}^{\infty}\psi_i = 1$. For Ψ to be a stationary distribution, there must exist a γ such that the following conditions hold:

$$\gamma\psi_0 = \pi_H \sum_{i=0}^{\infty} n(w_i,\theta_H)\psi_i \tag{21}$$

$$\gamma\psi_j = \pi_L n(w_{j-1},\theta_L)\psi_{j-1}, \quad j \geq 1 \tag{22}$$

Iterating on equation (22) implies the following:

$$\psi_m = \left(\frac{\pi_L}{\gamma}\right)^m n(w_{m-1},\theta_L)n(w_{m-2},\theta_L)\cdots n(w_0,\theta_L)\psi_0$$

Replacing in (21) implies the following equation:

$$\gamma = \pi_H \left(n(w_0, \theta_H) + \sum_{m=1}^{\infty} \left(\frac{\pi_L}{\gamma} \right)^m n(w_{m-1}, \theta_L) n(w_{m-2}, \theta_L) \cdots n(w_0, \theta_L) n(w_m, \theta_H) \right) \quad (23)$$

Given that in the original problem, we must have $n(w, \theta) \leq 1/b$. This means that the right hand side of the above equation is lower than $\sum_{m=0}^{\infty} (\pi_L/(b\gamma))^m$. Therefore, if we let $\gamma \to \infty$, the right hand side converges to a finite number, $\pi_H n(w_0, \theta_H)$. Notice that the left hand side is strictly increasing and the right hand side is strictly decreasing in γ. Moreover, at $\gamma = 0$ RHS is higher than LHS and at $\gamma = \infty$, RHS is lower than LHS. Because of this, if we knew that RHS was continuous, this would be sufficient to say that there is a γ satisfying equation (23) and that it is unique. To handle this last technical detail, we proceed as follows – Define γ_K as follows:

$$\gamma_K = \pi_H \left(n(w_0, \theta_H) + \sum_{m=1}^{K} \left(\frac{\pi_L}{\gamma_K} \right)^m n(w_{m-1}, \theta_L) n(w_{m-2}, \theta_L) \cdots n(w_0, \theta_L) n(w_m, \theta_H) \right).$$

By definition, $\gamma_K < \gamma_{K+1}$. We know that $n(w, \theta) < \frac{1}{b}$, therefore

$$\gamma_K < \pi_H \left(\frac{1}{b} + \sum_{m=1}^{K} \left(\frac{\pi_L}{\gamma_K} \right)^m \left(\frac{1}{b} \right)^{m+1} \right).$$

Suppose that $\frac{\pi_L}{b\gamma_K} < 1$ or $\frac{\pi_L}{b} < \gamma_K$. Then, the above inequality implies that

$$b\gamma_K < \pi_H \frac{1}{1 - \frac{\pi_L}{b\gamma_K}} \Rightarrow \gamma_K < \frac{\pi_H + \pi_L}{b} = \frac{1}{b}.$$

This shows that γ_K is a bounded increasing sequence. Hence, there exists γ^* such that $\gamma_K \to \gamma^*$ with $\gamma^* > \gamma_K$. It needs to be shown that at γ^*, RHS of (23) exists. Suppose not and that the sum is infinity. Define $F_K(\gamma)$ to be the RHS of (23) up to K-th term. $F_K(\gamma)$ is a continuous and decreasing function. Therefore, $\gamma_K = F_K(\gamma_K) > F_K(\gamma^*)$. Moreover, $F_K(\gamma^*) \to F(\gamma^*)$ and hence $F(\gamma^*) \leq \gamma^*$. This means that RHS of (23) cannot be infinity and (23) is satisfied at γ^*.

Now by Corollary 13, we know that

$$\exists\, A > 0 \quad ; \qquad n(w, \theta_H) \geq A n(w, \theta_L) \quad \forall w \in [\underline{w}, \overline{w}]. \quad (24)$$

Therefore, using (23), we will have

$$\begin{aligned}\gamma &= \pi_H\left(n(w_0,\theta_H)+\sum_{m=1}^{\infty}\left(\frac{\pi_L}{\gamma}\right)^m n(w_{m-1},\theta_L)n(w_{m-2},\theta_L)\cdots n(w_0,\theta_L)n(w_m,\theta_H)\right)\\ &\geq \pi_H A\sum_{m=0}^{\infty}\left(\frac{\pi_L}{\gamma}\right)^m n(w_m,\theta_L)n(w_{m-1},\theta_L)\cdots n(w_0,\theta_L).\end{aligned}$$

Hence

$$\sum_{m=0}^{\infty}\left(\frac{\pi_L}{\gamma}\right)^{m+1} n(w_m,\theta_L)n(w_{m-1},\theta_L)\cdots n(w_0,\theta_L)\leq\frac{\pi_L}{A\pi_H}.$$

Now define, ψ_0 as

$$\psi_0=\frac{1}{1+\sum_{m=0}^{\infty}\left(\frac{\pi_L}{\gamma}\right)^{m+1} n(w_m,\theta_L)n(w_{m-1},\theta_L)\cdots n(w_0,\theta_L)}$$

By the above inequality, we know that ψ_0 exists and it is greater than zero. Moreover, we can automatically define ψ_i's using (22). Hence, the definition of γ, being the solution to (23) together with the definition of ψ_0, makes sure that Ψ satisfies (21)-(22) and hence, it is a stationary distribution. As it appears in the proof, in some sense, bounded relative fertility[1] together with the resetting property at the top are the key elements of having a long-run stationary distribution. First, every time any one receives a high shock, her promised value is reset. Secondly, relatively, there are enough children being born by high types so that we get stationarity. Moreover, the above proof shows that when the set of w's is restricted to W, the stationary distribution is unique.[2]

8 Proofs

In this section we provide various proofs omitted in the paper.

8.1 Proof of Lemma 2

We show that when V is continuously differentiable and strictly convex, $v(w)$ is continuously differentiable, and strictly convex and $\eta v'(w)w - v(w)$ is strictly increasing. By definition, we have

$$V(N,W)=Nv(N^{-\eta}W)$$

[1] A high type's number of kids relative to the low type's

[2] A similar argument shows that there is a unique stationary distribution on $\mathcal{W}$ even if Assumption 12 does not hold. In this case, it follows that $\mathcal{W}$ is necessarily finite and the same logic applies.

Therefore, if V is continuously differentiable, v has the same property. Moreover, strict convexity and differentiability of V imply that $V_N(V_W)$ is strictly increasing in $N(W)$. Notice that

$$\begin{aligned} V_W(W,N) &= N^{1-\eta}v'(N^{-\eta}W) \\ V_N(W,N) &= v(N^{-\eta}W) - N^{-\eta}Wv(N^{-\eta}W) \end{aligned}$$

Hence, V_W being strictly increasing in W implies that $v'(w)$ is strictly increasing and hence $v(\cdot)$ is strictly convex. Moreover, V_N being strictly increasing in N implies that $v(w) - \eta wv'(w)$ is strictly decreasing in w.

8.2 Proof of Corollary 7

By Proposition 6, we know that

$$\lim_{w\to-\infty} w'(w,\theta_i) = \underline{w}_i$$

This implies that there exists a w_ϵ such that

$$\forall w \leq w_\epsilon,\ |w'(w,\theta_i) - \underline{w}_i| < \epsilon$$

By assumption 6, $\underline{w}_i < \bar{w}$. Now define,

$$\underline{w} = \min\left\{w_\epsilon, \underline{w}_1 - \epsilon, \underline{w}_2 - \epsilon, \cdots, \underline{w}_n - \epsilon, \inf_{w\in[w_\epsilon,\overline{w}],i} w'(w,\theta_i)\right\}$$

Notice that since w' is a continuous function that is always in R and the infimum is taken over a compact set, $\underline{\mathrm{w}}$ is well-defined. Pick $\epsilon > 0$ small enough so that $\inf_{w\in[w_\epsilon,\overline{w}],i} w'(w,\theta_i) < \underline{w}_j - \epsilon$ for all j. Then by definition of w_ϵ we must have

$$w'(w,\theta_i) \in [\underline{w},\bar{w}],\ \forall w \in [\underline{w},\bar{w}]$$

■

Since utility is unbounded below and η is negative, $n(w,\theta_i)$ must be positive. Hence we can have the following corollary:

Corollary 13 *For all $w \in [\underline{w},\bar{w}]$, we must have $n(w,\theta_i) \geq \underline{n}$ and $\frac{n(w,\theta_n)}{n(w,\theta_i)} \geq A$, for all $i \in \{1,\cdots,n\}$ and for some $\underline{n}, A > 0$.*

8.3 Proof of Remark 10

Since $l(w,\theta_n) > 0$ for all $w \in [\underline{w}, w_0]$, resetting property at the top holds. Therefore, by definition

$$\begin{aligned}
\gamma(\Psi) \cdot \Psi\left(\{w_0\}\right) &= \pi_n \int_S n(w,\theta_n) d\Psi(w) \\
\gamma(\Psi) &= \int_S \sum_{i=1}^{n} \pi_i n(w,\theta_i) d\Psi(w) \\
&\leq \pi_n \int_S n(w,\theta_n) d\Psi(w) \\
&\quad +(1-\pi_n)A^{-1} \int_S n(w,\theta_n) d\Psi(w) \\
&= \left(\pi_n + (1-\pi_n)A^{-1}\right) \int_S n(w,\theta_n) d\Psi(w)
\end{aligned}$$

Therefore,

$$\Psi\left(\{w_0\}\right) \geq \frac{\pi_n A}{1-\pi_n+\pi_n A}$$

■

9 A Numerical Example

In this section, we provide a numerical solution for the model provided in section 3 of the paper. In light of this example, we are able to provide some more intuitive properties of the model that we have not proved in the paper.

We assume that individuals have CRRA preferences over consumption and leisure

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}, \text{ and } h(m) = \phi \frac{m^{1-\sigma}}{1-\gamma}$$

in which $m = 1 - l - bn$ is leisure, l is hours worked and n is number of kids for each parent.

For this example we assume the following values for parameters: $\beta = 0.3$, $R = 4$, $\sigma = 1.5$, $\phi = 0.5$, $b = 0.41$ and $\eta = -2$. We assume two levels of productivity shocks $\{\theta_L, \theta_H\} = \{2, 6\}$. Shocks are i.i.d across generations and dynasties and the probability of the high shock is $\pi_H = 0.1$.

Figure 2, presents the optimal policy functions in the recursive problem given above specification. It can be seen that whenever hours worked are positive, the per capita continuation value of parents with high productivity shock is constant, i.e., the Resetting Property. Moreover, as it is established in proposition 6 in the paper, the per capita continuation value for a parent with a low productivity shock converges to a finite number as parent's promised utility tends to $-\infty$.

One observation that cannot be shown analytically is that fertility for less productive agents is lower than that of more productive agents. When there is no private information, full risk sharing and cost minimization deliver this result. Full risk sharing implies that per capita marginal utility from having children must be equated across types. This implies that consumption per child is negatively correlated with the number of children. On the other hand, kids are more expensive for more productive types (in terms of their time). Therefore, it is cost effective to deliver utility from having children to more productive types by giving more per capita consumption to each of their children and have them have fewer children. With private information, however, full risk sharing is violated and the argument above does not work. However, it is observed in our numerical example that more productive types have a lower fertility rate. Our calculations, also show that the spread in fertility is lower under private information compared to full information.

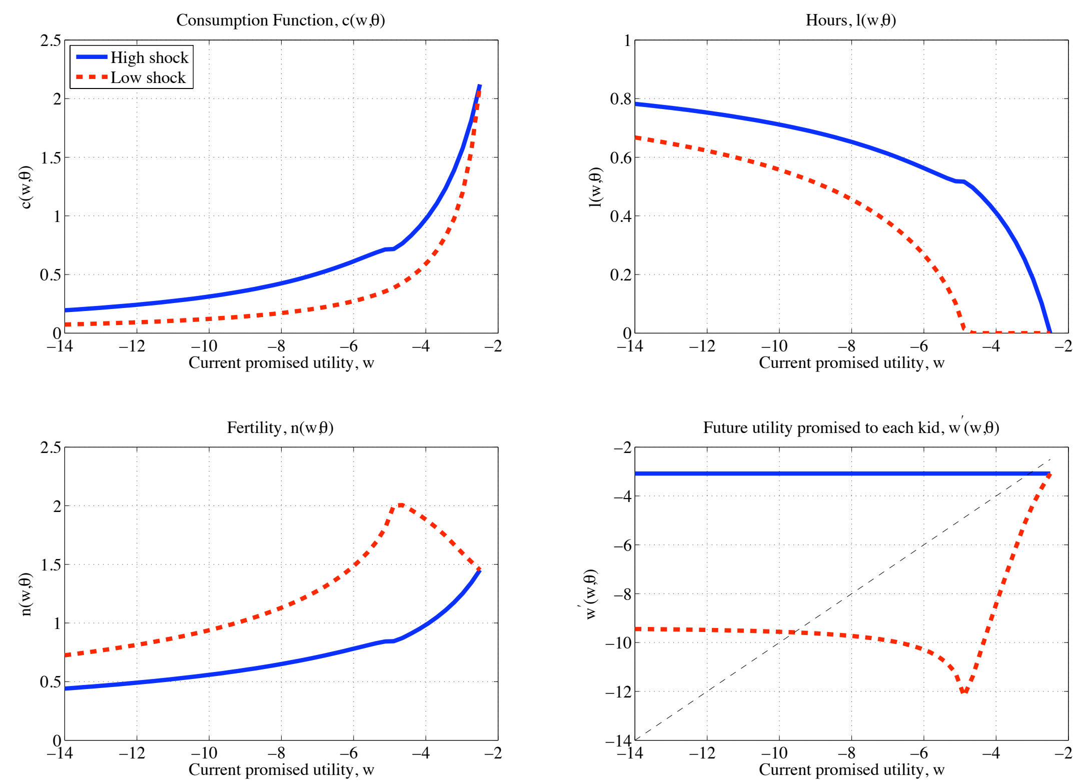

Figure 2: Optimal consumption, hours, fertility and promised utility allocations for the numerical example

It is important to note here that incentives are provided both by the level of per capita promised utility to the children and the number of children. In other words the future utility that is promised to a parent is $n(w,\theta)^{\eta}w'(w,\theta)$. This promised utility is always monotone increasing in the current utility promised to the parent. This property is similar to benchmark dynamic Mirrleesian models with no fertility choice. Figure 3 illustrates this.

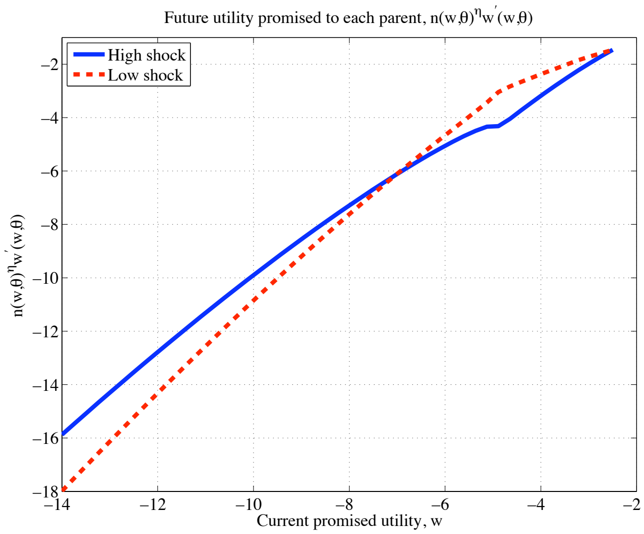

Figure 3: Future promised utility to the parents.

We can use the construction procedure in section 7 to calculate the stationary distribution of per capita promised utilities and the growth rate of population, γ_{Ψ^*}. The stationary distribution is shown in Figure 4. For this economy, the growth rate of population is $\gamma_{\Psi^*} = 1.0355$. As it is mentioned in remark 10 in the paper, a positive fraction of agents are always at the resetting value $w_0 = -3.0826$.

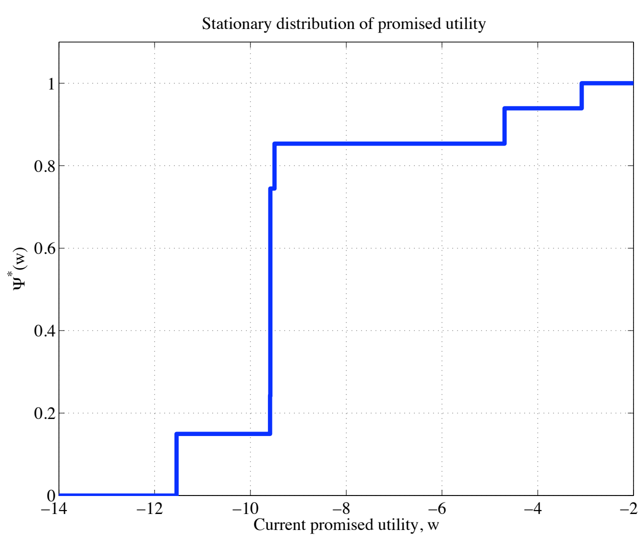

Figure 4: Stationary distribution

10 Linear Utility of Leisure

In this section we focus on the case with linear utility from leisure. This allows to prove many of the unproved assumptions in the main body of the paper for the general case. To do so, we consider the problem (P1) in the paper with the additional assumption that $h(m) = \psi m$:

$$V(N,W) = \min_{C_i,L_i,N_i,W_i'} \quad \sum_{i=1}^{I} \pi(\theta_i)\left[C_i - \theta_i L_i + \frac{1}{R}V(N_i,W_i')\right] \qquad (P5)$$

$$\text{s.t} \quad \sum_{i=1}^{I} \pi(\theta_i)\left[N^\eta\left(u\left(\frac{C_i}{N}\right) + \psi\left(1 - \frac{L_i}{N} - b\frac{N_i}{N}\right)\right) + \beta W_i'\right] \geq W$$

$$N^\eta\left(u\left(\frac{C_i}{N}\right) + \psi\left(1 - \frac{L_i}{N} - b\frac{N_i'}{N}\right)\right) + \beta W_i' \geq$$

$$N^\eta\left(u\left(\frac{C_j}{N}\right) + \psi\left(1 - \frac{\theta_j L_j}{\theta_i N} - b\frac{N_j'}{N}\right)\right) + \beta W_j'$$

$$\forall i,j.$$

Notice that the set of reports is not restricted to lower reports since we can prove that general incentive compatibility is equivalent to local downward constraints being binding and output being increasing, a similar approach to Thomas and Worrall (1990). Notice that adding the IC constraint where j pretends to be i and the reverse implies that:

$$\frac{\theta_i L_i}{\theta_j N} + \frac{\theta_j L_j}{\theta_i N} \geq \frac{L_i}{N} + \frac{L_j}{N}.$$

Therefore, if $\theta_i > \theta_j$, then $\theta_i L_i \geq \theta_j L_j$ which means output is increasing.
Moreover, if we assume that local downward IC constraints are binding and output is increasing, it can be easily shown that the local upward constraints are satisfied. Thus, summing over local incentive constraints gives the general ones. We also assume that output being increasing is not binding so we can neglect it. Therefore the functional equation becomes

the following:

$$
\begin{aligned}
V(N,W) = \min_{C_i,L_i,N_i',W_i'} \quad & \sum_{i=1}^{I} \pi(\theta_i)\left[C_i - \theta_i L_i + \frac{1}{R} V(N_i', W_i')\right] \\
\text{s.t} \quad & \sum_{i=1}^{I} \pi(\theta_i)\left[N^\eta \left(u\left(\frac{C_i}{N}\right) + \psi\left(1 - \frac{L_i}{N} - b\frac{N_i'}{N}\right)\right) + \beta W_i'\right] \geq W \\
& N^\eta \left(u\left(\frac{C_i}{N}\right) + \psi\left(1 - \frac{L_i}{N} - b\frac{N_i'}{N}\right)\right) + \beta W_i' \geq \\
& \quad N^\eta \left(u\left(\frac{C_{i-1}}{N}\right) + \psi\left(1 - \frac{\theta_{i-1} L_{i-1}}{\theta_i N} - b\frac{N_{i-1}'}{N}\right)\right) + \beta W_{i-1}'.
\end{aligned}
$$

Let $-\lambda N^{1-\eta}$ be the lagrange multiplier on promise-keeping constraint and $-\mu_i N^{1-\eta}$ be the multiplier for i-th IC constraint. Then the first order condition for hours worked is the following:

$$
\begin{aligned}
\pi_I \theta_I &= (\lambda \pi_I + \mu_I)\psi \\
\pi_i \theta_i &= (\lambda \pi_i + \mu_i - \frac{\theta_i \mu_{i+1}}{\theta_{i+1}})\psi, \quad i = 2, \cdots, n \qquad (25) \\
\pi_1 \theta_1 &= (\lambda \pi_1 - \frac{\theta_1 \mu_2}{\theta_2})\psi. \qquad (26)
\end{aligned}
$$

We can define $\mu_1 = \mu_{I+1} = 0$ and (25) holds for $i = 1, ..., I$. If we divide the i-th equation by θ_i and sum over all i's, the μ_i's will cancel and we have

$$
\lambda = \frac{1}{\psi \sum_i \frac{\pi_i}{\theta_i}} = \frac{1}{\psi \mathbb{E}\frac{1}{\theta}} \qquad (27)
$$

Therefore,

$$
\mu_i = \frac{1}{\psi}\theta_i \sum_{j \geq i} \pi_j - \frac{\theta_i \sum_{j\geq i} \frac{\pi_j}{\theta_j}}{\psi \sum_j \frac{\pi_j}{\theta_j}} = \theta_i \frac{\sum_j \frac{\pi_j}{\theta_j} \sum_{j \geq i} \pi_j - \sum_{j \geq i} \frac{\pi_j}{\theta_j}}{\psi \sum_j \frac{\pi_j}{\theta_j}}.
$$

Since θ_i's increasing, all the μ_i's are positive.
The first order conditions with respect to consumption are:

$$
\pi_i = (\lambda \pi_i + \mu_i - \mu_{i+1}) u'\left(\frac{C_i}{N}\right).
$$

Obviously, we need consumption to be increasing and marginal utility to be positive. This gives us a condition on distribution of θ_i. Moreover, we can see that consumption is independent of the state variable (N, W).

The first order conditions with respect to N_i', W_i' are:

$$\pi_i \frac{1}{R} V_N(N_i', W_i') = -b(\lambda\pi_i + \mu_i - \mu_{i+1})\psi \tag{28}$$
$$\pi_i \frac{1}{R} V_W(N_i', W_i') = N^{1-\eta}(\lambda\pi_i + \mu_i - \mu_{i+1})\beta. \tag{29}$$

Now for every i, define *after-tax-productivity* as follows:

$$\tilde{\theta}_i = \psi \frac{\lambda\pi_i + \mu_i - \mu_{i+1}}{\pi_i}.$$

Notice that we have $u'(C_i/N)\tilde{\theta}_i = \psi$ and $\tilde{\theta}_i$ does not depend on the state variables. From before, we know that there exists a function $v(\cdot)$ such that $V(N,W) = Nv(N^{-\eta}W)$. Therefore,

$$V_N(N,W) = v(w) - \eta w v'(w), V_W(N,W) = N^{1-\eta}v'(w)$$

where $w = N^{-\eta}W$. Hence, from (29) we have that:

$$\eta w_i' v'(w_i') - v(w_i') = bR\tilde{\theta}_i$$
$$N_i'^{1-\eta} v'(w_i) = \beta R N^{1-\eta}\tilde{\theta}_i.$$

The above, implies that $n_i = N_i'/N, w_i'$ are also independent of the state. Moreover, from the Envelope condition we have that:

$$V_W(N,W) = \lambda N^{1-\eta} = \frac{N^{1-\eta}}{\psi \sum_i \frac{\pi_i}{\theta_i}} = N^{1-\eta}v'(w).$$

Therefore, $v(\cdot)$ is a linear function and we have:

$$v(w) = A + \frac{w}{\psi \sum_i \frac{\pi_i}{\theta_i}} \Rightarrow V(N,W) = AN + \frac{WN^{1-\eta}}{\psi \sum_i \frac{\pi_i}{\theta_i}}.$$

Notice that for the problem to be concave, we need $N^\eta h(\frac{M}{N})$ to be concave and therefore, to have $N^{\eta-1}M$ be weakly concave, we must have $\eta = 1$. In this case $V(N,W)$ is linear in (N,W) and therefore weakly convex.

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