Bid–Ask Equilibria and the Short-Sale Restriction

Supplemental Appendix to "Adverse Selection and Market Structure in Annuity and Life Insurance with Public Annuitization"

Roozbeh Hosseini*
University of Georgia

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This appendix asks whether the model of the main text admits an equilibrium with a bid–ask spread once the no-short-sale restriction is relaxed, in the spirit of Bisin and Gottardi (1999). The answer is that it does not. I show that in any equilibrium each mortality-contingent claim is traded on at most one side, so a bid–ask spread is never made operative by two-sided trade, and every equilibrium coincides with an equilibrium of the buy-only economy analyzed in the main text. In this environment the short-sale restriction is therefore an expositional device, not a substantive assumption.

Throughout, section and result numbers without a qualifier refer to this appendix. I flag references to the paper as "the main text."

1 Setup

The primitive economy is the two-period model of Section 2 of the main text. I recall only what is needed. There is a continuum of households indexed by their survival probability $\pi \in [\underline{\pi}, \overline{\pi}]$, distributed according to μ. A household of type π has preferences

$$u(c_1) + \beta\pi U(c_2) + \beta(1-\pi)v(b)$$

over first-period consumption c_1, survival-state consumption c_2, and bequest b. The functions u, U, and v are twice continuously differentiable, strictly increasing, strictly concave, and satisfy the Inada conditions. Each household is endowed with e units of the good in period 1. A riskless technology turns one unit of the period-1 good into $A > 1$ units of the period-2 good, and saving $s \geq 0$ cannot be shorted. Social Security taxes the endowment at rate τ and pays a transfer T to survivors in period 2.

Two contingent claims trade. One unit of the *annuity* pays one unit of the good in the survival state. One unit of *life insurance* pays one unit of the good in the death state. In the main text a household may buy but not short either claim, so that $a \geq 0$ and $l \geq 0$, and

*Department of Economics, Terry College of Business, University of Georgia, 620 South Lumpkin Street, Athens, GA 30602. Email: roozbeh@uga.edu

each claim trades at a single price. Here I drop the short-sale restriction and let each claim be held in any quantity, $a, l \in \mathbb{R}$.

Bid–ask prices. The annuity is bought at the *ask* p^a_+ and sold at the *bid* p^a_-, with $p^a_- \leq p^a_+$. Life insurance is bought at the ask p^l_+ and sold at the bid p^l_-, with $p^l_- \leq p^l_+$. The cost of an annuity position a is

$$P^a(a) = \begin{cases} p^a_+ \, a & \text{if } a \geq 0 \\ p^a_- \, a & \text{if } a < 0, \end{cases}$$

and the cost of a life-insurance position l is

$$P^l(l) = \begin{cases} p^l_+ \, l & \text{if } l \geq 0 \\ p^l_- \, l & \text{if } l < 0. \end{cases}$$

A short annuity position $a < 0$ raises period-1 resources by $-p^a_- a = p^a_- |a|$ in exchange for the obligation to pay $|a|$ in the survival state. A short life-insurance position $l < 0$ raises $-p^l_- l = p^l_- |l|$ in exchange for the obligation to pay $|l|$ in the death state.

A household of type π solves

$$\max_{c_1, c_2, b, a, l, s} \; u(c_1) + \beta \pi U(c_2) + \beta(1-\pi) v(b) \tag{1}$$

subject to

$$\begin{aligned} c_1 + P^a(a) + P^l(l) + s &= e(1-\tau) \\ c_2 &= As + a + T \\ b &= As + l \\ c_1, c_2, b \geq 0, \quad s &\geq 0, \quad a, l \in \mathbb{R}. \end{aligned}$$

Pricing. Each side of each market is a competitive, zero-profit pool. Contributions are invested at the gross return A and fund the pool's state-contingent payouts. For the annuity, the ask is set by the pool of buyers, $\{\pi : a(\pi) > 0\}$, and the bid by the pool of sellers, $\{\pi : a(\pi) < 0\}$:

$$A p^a_+ \int_{\{a(\pi)>0\}} a(\pi)\, d\mu(\pi) = \int_{\{a(\pi)>0\}} \pi \, a(\pi)\, d\mu(\pi), \tag{2}$$

$$A p^a_- \int_{\{a(\pi)<0\}} \big(-a(\pi)\big)\, d\mu(\pi) = \int_{\{a(\pi)<0\}} \pi \, \big(-a(\pi)\big)\, d\mu(\pi). \tag{3}$$

For life insurance the payout in the death state accrues at rate $1-\pi$, so the ask is set by the buyers, $\{\pi : l(\pi) > 0\}$, and the bid by the sellers, $\{\pi : l(\pi) < 0\}$:

$$A p^l_+ \int_{\{l(\pi)>0\}} l(\pi)\, d\mu(\pi) = \int_{\{l(\pi)>0\}} (1-\pi)\, l(\pi)\, d\mu(\pi), \tag{4}$$

$$A p^l_- \int_{\{l(\pi)<0\}} \big(-l(\pi)\big)\, d\mu(\pi) = \int_{\{l(\pi)<0\}} (1-\pi)\, \big(-l(\pi)\big)\, d\mu(\pi). \tag{5}$$

Each price is thus the position-weighted average payout rate of the households on that side of that market.

A market *side* is *active* if it carries positive aggregate trade. A bid–ask spread is *operative* if some claim is traded on *both* sides, so that both its bid and its ask are pinned down by trade. If a claim is traded on only one side, the opposite quote is not disciplined by any trade and the active side is just a single buy-only price.

Two properties from the main text. I use two structural properties of the buy-only economy. Each carries over to (1) unchanged, because once the relevant side's price is fixed the household problem is the one solved in the main text.

(P1) *Continuity and monotonicity.* Demand is continuous in π. This follows immediately from the maximum theorem: the survival probability enters only the objective in (1), and it does so continuously; the budget set is compact at the prices under consideration and does not depend on π, so it is trivially continuous in π; and strict concavity of u, U, and v makes the maximizer unique. The optimal allocation is therefore a continuous function of π. Conditional on which side of each market is active, the household faces a single price for each claim and the problem is the buy-only problem of the main text; the monotonicity argument given there then applies unchanged, so the annuity position $a(\pi)$ is weakly increasing and the life-insurance position $l(\pi)$ weakly decreasing in π (Lemma on the cutoff structure, main text, Section 3).

(P2) *Single-signed net annuity at fair prices.* At the actuarially fair prices $p^a(\pi) = \pi/A$ and $p^l(\pi) = (1-\pi)/A$, the net annuity purchase $a(\pi) - l(\pi)$ has the same sign for every type $\pi \in [\underline{\pi}, \overline{\pi}]$ (Proposition 1 of the main text).

2 Net Annuity and the Redundancy of the Two Contracts

I first record that, exactly as in the buy-only economy, only the net annuity position matters. Combining the two period-2 constraints in (1), $c_2 = As + a + T$ and $b = As + l$, and subtracting gives

$$a(\pi) - l(\pi) = c_2(\pi) - b(\pi) - T. \tag{6}$$

This is the *net annuity purchase* of the main text: the excess of survival-state resources over death-state resources, net of the public annuity T. A household is *long survival* when $a - l > 0$ and *long death* when $a - l < 0$.

The next lemma says that the riskless asset makes the two contracts redundant: each long position can be reached either by buying one claim or by selling the other.

Lemma 1 (Redundancy). *For any household, selling one unit of life insurance and saving* $1/A$ *yields exactly the payoff of buying one unit of the annuity. Symmetrically, selling one unit of the annuity and saving* $1/A$ *yields exactly the payoff of buying one unit of life insurance.*

Proof. Write a trade as its payoff in the two states, (survival, death). The four contract trades and the saving trade are

$$\text{buy annuity} = (1, 0), \quad \text{sell annuity} = (-1, 0),$$

$$\text{buy LI} = (0, 1), \quad \text{sell LI} = (0, -1), \quad \text{save } 1/A = (1, 1).$$

Adding,

$$(0, -1) + (1, 1) = (1, 0) \quad \text{and} \quad (-1, 0) + (1, 1) = (0, 1),$$

which are the two claims of the lemma.

The same point reads directly off the budget set. Replace (l, s) in (1) by $(l-1,\ s+1/A)$. The bequest

$$b = As + l = A\Big(s + \tfrac{1}{A}\Big) + (l - 1)$$

is unchanged, while survival-state resources $c_2 = As + a + T$ rise by one unit. This is the same change in the budget set as replacing a by $a + 1$. □

The two ways of taking a long-survival position differ only in cost. Buying the annuity costs p^a_+ per unit. Selling life insurance and saving $1/A$ costs $\frac{1}{A} - p^l_-$ per unit. Because prices are anonymous and linear, this comparison is the same for every household. The next proposition draws out the consequence.

Proposition 1 (The two long positions are one market)**.** *In any equilibrium the long-annuity pool and the short–life-insurance pool cannot both be active as distinct markets. Either at most one of them carries positive trade, or they trade at prices satisfying*

$$p^a_+ = \frac{1}{A} - p^l_-, \tag{7}$$

in which case they draw the same households,

$$\frac{\int_{\{a(\pi)>0\}} \pi\, a(\pi)\, d\mu(\pi)}{\int_{\{a(\pi)>0\}} a(\pi)\, d\mu(\pi)} = \frac{\int_{\{l(\pi)<0\}} \pi\, \big(-l(\pi)\big)\, d\mu(\pi)}{\int_{\{l(\pi)<0\}} \big(-l(\pi)\big)\, d\mu(\pi)}, \tag{8}$$

and constitute a single market. The symmetric statement holds for the long–life-insurance pool and the short-annuity pool.

Proof. Every household faces the same unit cost of a long-survival position: p^a_+ through the annuity ask, and $\frac{1}{A} - p^l_-$ through the life-insurance bid (Lemma 1).

Suppose $p^a_+ < \frac{1}{A} - p^l_-$. Then the annuity is the strictly cheaper way to go long survival, so no household sells life insurance for this purpose, and the short–life-insurance pool is empty.

Suppose $p^a_+ > \frac{1}{A} - p^l_-$. Then selling life insurance is the strictly cheaper way to go long survival, so no household buys the annuity, and the long-annuity pool is empty.

Active coexistence therefore requires

$$p^a_+ = \frac{1}{A} - p^l_-,$$

which is (7). From the pricing equations (2) and (5),

$$Ap^a_+ = \frac{\int_{\{a(\pi)>0\}} \pi\, a(\pi)\, d\mu(\pi)}{\int_{\{a(\pi)>0\}} a(\pi)\, d\mu(\pi)}, \qquad Ap^l_- = \frac{\int_{\{l(\pi)<0\}} (1-\pi)\,\big(-l(\pi)\big)\, d\mu(\pi)}{\int_{\{l(\pi)<0\}} \big(-l(\pi)\big)\, d\mu(\pi)}.$$

Substituting these into (7) and using $A\big(\frac{1}{A} - p^l_-\big) = 1 - Ap^l_-$,

$$\frac{\int_{\{a(\pi)>0\}} \pi\, a(\pi)\, d\mu(\pi)}{\int_{\{a(\pi)>0\}} a(\pi)\, d\mu(\pi)} = 1 - \frac{\int_{\{l(\pi)<0\}} (1-\pi)\,\big(-l(\pi)\big)\, d\mu(\pi)}{\int_{\{l(\pi)<0\}} \big(-l(\pi)\big)\, d\mu(\pi)} = \frac{\int_{\{l(\pi)<0\}} \pi\,\big(-l(\pi)\big)\, d\mu(\pi)}{\int_{\{l(\pi)<0\}} \big(-l(\pi)\big)\, d\mu(\pi)},$$

which is (8): the long-annuity pool and the short–life-insurance pool have the same position-weighted average survival, so they draw the same households. At the common cost (7) the split of a household's long-survival position between the two contracts is indeterminate and payoff-irrelevant. The two pools are one market. □

Proposition 1 is the bid–ask counterpart of the main text's focus on the net annuity position. Gross holdings of the two contracts are not separately determined; only the net position $a - l$ is. "Buying an annuity" and "selling life insurance" are two names for the same trade—taking a long-survival position—and adverse selection prices them off the same pool of households.

3 No Equilibrium Has an Operative Bid–Ask Spread

Theorem 1. *In any equilibrium, each mortality-contingent claim is traded on at most one side. Consequently no equilibrium has an operative bid–ask spread, and every equilibrium coincides with an equilibrium of the buy-only economy of the main text.*

Proof. Suppose, toward a contradiction, that the annuity is traded on both sides, so that both $\{\pi : a(\pi) > 0\}$ and $\{\pi : a(\pi) < 0\}$ have positive μ-measure.

By monotonicity (P1), $a(\pi)$ is weakly increasing, so the two sets are separated by cutoffs $\pi_- \le \pi_+$ with

$$a(\pi) > 0 \ \text{ for } \pi > \pi_+, \qquad a(\pi) < 0 \ \text{ for } \pi < \pi_-.$$

Every buyer therefore has survival probability above π_+, and every seller has survival probability below π_-. Comparing the ask and the bid through the pricing equations (2) and (3),

$$Ap^a_+ = \frac{\int_{\{a(\pi)>0\}} \pi\, a(\pi)\, d\mu(\pi)}{\int_{\{a(\pi)>0\}} a(\pi)\, d\mu(\pi)} \ \ge\ \pi_+,$$

$$Ap^a_- = \frac{\int_{\{a(\pi)<0\}} \pi\,\big(-a(\pi)\big)\, d\mu(\pi)}{\int_{\{a(\pi)<0\}} \big(-a(\pi)\big)\, d\mu(\pi)} \ \le\ \pi_-.$$

Since $\pi_- \le \pi_+$, the bid lies weakly below the ask.

Now consider the type

$$\hat{\pi} := Ap^a_+ \ \ge\ \pi_+.$$

This type belongs to the long-annuity pool, so it is an annuity buyer, and it faces the actuarially fair ask $p^a_+ = \hat{\pi}/A$. It funds its bequest through saving, whose fair gross return A values one unit of the death claim at $(1-\hat{\pi})/A$. Hence type $\hat{\pi}$ faces fair prices on every margin it uses, and the budget set in (1) reduces to the full-information budget of type $\hat{\pi}$. As a strict buyer it has positive net annuity,

$$a(\hat{\pi}) - l(\hat{\pi}) > 0.$$

Consider next the type

$$\tilde{\pi} := Ap^a_- \;\leq\; \pi_-.$$

This type belongs to the short-annuity pool, so it is an annuity seller, and it faces the actuarially fair bid $p^a_- = \tilde{\pi}/A$. By the same reduction, it faces fair prices on every margin it uses. As a strict seller it has negative net annuity,

$$a(\tilde{\pi}) - l(\tilde{\pi}) < 0.$$

Both $\hat{\pi}$ and $\tilde{\pi}$ face actuarially fair prices. By the single-sign property (P2) their net annuity purchases must have the same sign. This contradicts

$$a(\hat{\pi}) - l(\hat{\pi}) > 0 > a(\tilde{\pi}) - l(\tilde{\pi}).$$

Therefore the annuity is not traded on both sides.

The identical argument, with π replaced by $1-\pi$ throughout the life-insurance pricing equations (4) and (5), rules out two-sided life-insurance trade.

With at most one side of each claim active, no quote is disciplined by opposing trade, so no bid–ask spread is operative. Each active side is a buy-only position, priced exactly as in the main text. □

Remark (Why the spread cannot form). A bid–ask spread, in the sense of Bisin and Gottardi (1999), is the device that separates the buyers of a contract from its sellers. In the present economy adverse selection has *already* sorted households monotonically by survival probability, so within any single contract the would-be buyers are the high-survival types and the would-be sellers the low-survival types. Property (P2) says that households facing fair prices cannot want net positions of opposite sign. The two-sided trade that an operative spread requires is therefore impossible: what the spread would separate, the sorting of demand has already separated.

Remark (Relation to Theorem 1 of the main text). Theorem 1 leaves at most one active side in each contract, and Proposition 1 identifies the long-annuity side with the short–life-insurance side (and the long–life-insurance side with the short-annuity side). Combining the two, at most one mortality-contingent position—long survival or long death—is traded across the two contracts, and it is traded on its buy side. This is exactly the conclusion of Theorem 1 of the main text: at most one private insurance market is active, now obtained without the short-sale restriction. Allowing short sales and bid–ask pricing changes neither the set of equilibrium allocations nor the market-structure result, so the short-sale restriction of the main text is without loss of generality for these conclusions.

References

[1] Bisin, A. and P. Gottardi (1999), "Competitive Equilibria with Asymmetric Information," *Journal of Economic Theory* 87(1), 1–48.