Adverse Selection, Public Annuitization, and the Structure of Annuity and Life Insurance Markets*

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Abstract

This paper studies a competitive insurance economy with private information about survival probabilities. Households value consumption if they survive and bequests if they do not. They trade annuity and life insurance contracts and save through a risk-free technology. Social Security provides public annuitization. Contracts are nonexclusive, in the sense that insurers cannot condition on the household's full portfolio, and households can buy but not short insurance contracts. In this environment competitive equilibrium exists. At most one private insurance market is active. Larger Social Security benefits reduce private annuity demand, shift annuity pools toward longer-lived households, and raise annuity prices. For sufficiently large benefits, life insurance rather than annuities is the active private market. The model also delivers an ex ante efficient allocation that can be implemented by a tax-transfer policy.

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1 Introduction

Annuities and life insurance insure the same event from opposite sides. An annuity pays if the household survives. Life insurance pays if the household dies. In the data these two markets differ greatly. Private annuity holdings are small. Hosseini (2015), Pashchenko (2013), and Johnson et al. (2004) report that only a small fraction of retirees hold private annuities in their own names in the United States. At the same time, life insurance is a very large market. Cawley and Philipson (1999) describe it as the largest private insurance market. According to American Council of Life Insurers (2024), total life insurance in force in the United States was $22.2 trillion at year-end 2023, and more than 134 million individual life insurance policies were in force. A theory of insurance with private information about mortality should account for both facts.

There is also evidence that private annuity markets are subject to selection. Friedman and Warshawsky (1990) and Mitchell et al. (1999) show that private annuities are priced at rates above those implied by average population mortality. Finkelstein and Poterba (2004) find evidence consistent with adverse selection in the U.K. annuity market. These findings suggest a simple question. If mortality risk is privately known, and if households care both about consumption in old age and resources left to heirs, what structure should private insurance markets take?

This paper studies a two-period economy in which households know their own survival probabilities. They value consumption if alive in period 2 and bequests if not. They can buy annuities, buy life insurance, and save. Insurance is provided through competitive pools. Social Security taxes households in period 1 and pays benefits to survivors in period 2. The central question is how private mortality information shapes the joint equilibrium of annuity and life insurance markets, and how that equilibrium changes with public annuitization.

The model has three main results. First, competitive equilibrium exists. Second, at most one private insurance market is active. If annuities trade, life insurance does not. If life insurance trades, annuities do not. Third, larger Social Security benefits reduce private annuity demand, raise annuity prices, and shift activity toward life insurance. These results follow from the interaction of private information, bequest motives, and nonexclusive contracting.

The "one active market" result is the central implication of the model. In the data the contrast between a thin annuity market and a very large life-insurance market is not absolute, but it is pronounced. The model offers a simple account of this asymmetry. Households with higher survival probabilities are willing to pay more for annuities. Households with lower survival probabilities are willing to pay more for life insurance. Once prices reflect the composition of each pool, however, only one private market remains open in equilibrium.

Public annuitization then matters not only because it changes the amount of private annuity trade, but because it changes which private insurance market can operate.

The nonexclusive-contracting assumption is simple. Insurers observe the contracts they sell, but not the household's entire portfolio. They cannot condition terms on other trades. Households can buy but not short either insurance contract. With these restrictions, insurers face the composition of each pool as a market object rather than a contractual choice variable. This is the setting in which equilibrium exists and can be characterized.

The policy problem enters naturally because Social Security is a publicly provided annuity. In the model, a larger Social Security program lowers the demand for private annuities. This is partly a direct substitution effect. It is also a selection effect. Lower-survival households leave the annuity pool first, and this raises the annuity price faced by those who remain. The same policy increases the relative attractiveness of life insurance by making the survival state better insured. The paper also characterizes an ex ante efficient allocation and the tax-transfer rule that implements it.

The paper is related to several strands of literature. Yaari (1965) provides the classic benchmark for annuitization. Work on annuity markets with adverse selection includes Abel (1986), Friedman and Warshawsky (1990), Mitchell et al. (1999), and Finkelstein and Poterba (2004). Davidoff et al. (2005) study the welfare value of annuities, and Pashchenko (2013) studies the low rate of private annuitization.

The paper is also related to work linking Social Security and annuity markets. Walliser (2000) studies adverse selection in annuity markets and the effects of Social Security privatization. Hosseini (2015) conducts a quantitative welfare analysis of Social Security in an annuity market with adverse selection. Villeneuve (2003) and Hong and Rios-Rull (2007) study pensions together with private insurance markets, including life insurance and annuities. The present paper differs from this literature in the object it characterizes. The focus here is the competitive equilibrium of a nonexclusive insurance market with private mortality information. In this setting equilibrium exists, pooling prices are determined jointly with household participation, and the main theorem shows that at most one private insurance market is active. The comparative statics then describe how public annuitization shifts the economy across these market structures.

Finally, the paper uses the competitive framework for nonexclusive trade developed in Bisin and Gottardi (1999), Bisin and Gottardi (2003), Dubey and Geanakoplos (2002), and Attar et al. (2014). The present paper combines these elements in a model with both annuity and life insurance trade and uses the model to characterize the structure of equilibrium insurance markets.

Section 2 describes the economy and defines equilibrium. Section 3 derives the main equilibrium properties and proves existence. Section 4 studies Social Security, prices, and policy. Section 5 concludes.

2 The Economy

There is a two-period economy with one consumption good. The good can be consumed in period 1, consumed in period 2, or left as a bequest. Households differ in their probability of surviving to period 2. A household of type $\pi \in [\underline{\pi}, \overline{\pi}]$ survives with probability π. Types are distributed according to μ. There is a continuum of households and a competitive fringe of intermediaries.

Information. There is no aggregate uncertainty. Lifetime risk is idiosyncratic. A household knows its own survival probability at the beginning of period 1, before trade takes place. The distribution of types is common knowledge, but type itself is private information. Survival or death in period 2 is publicly observed.

Consumers. Each household is endowed with e units of the consumption good in period 1. Preferences are preferences over first period consumption, c_1, second period consumption (if they are alive in the second period), c_2, and the bequest left if they die, b. Lifetime expected utility is

$$u(c_1) + \beta\pi U(c_2) + \beta(1-\pi)v(b)$$

where u, U, and v satisfy the following assumption.

Assumption 1 *$u(\cdot)$, $U(\cdot)$ and $v(\cdot)$ are twice continuously differentiable, strictly increasing, strictly concave and satisfy INADA conditions ($\lim_{c\to 0} u'(c) = \infty$, etc).*

Households discount period-2 utility at rate $0 < \beta < 1$.

Contracts. There are two insurance contracts. One unit of annuity pays one unit of the consumption good if the household is alive in period 2. One unit of life insurance pays one unit if the household dies[1]. Households can buy, but not short, either contract. Buying a contract is equivalent to buying a share in an insurance pool. Contributions are made in period 1 and payouts are state contingent in period 2.

Technology. There is a risk-free saving technology with gross return $A > 1$ available to all agents.

[1] In the background, the payment in the death state accrues to heirs, who are not modeled explicitly.

Intermediaries. Intermediaries operate annuity and life-insurance pools. Given expected per-capita contributions and payouts in each pool, an intermediary chooses pool size. Contributions received in period 1 are invested in the saving technology.

Prices and Nonexclusive Trade. Annuity and life-insurance contracts trade at prices p^a and p^l. Intermediaries observe the contracts they sell, but they do not observe the household's full portfolio or other trades. Contract terms therefore cannot be conditioned on total insurance holdings or asset positions. All households face the same market price for a given contract, independent of purchase volume or portfolio composition.

Contracts are linear: one unit of annuity or life insurance is purchased at a constant unit price. The model takes this contract form as given through the pooling structure rather than deriving it from a richer strategic model of insurer competition.[2]

Social Security. Social Security taxes households at rate τ in period 1 and pays a lump-sum transfer to survivors in period 2. It is therefore a publicly provided annuity. The government transfers Social Security receipts across periods using the same saving technology.

Given prices (p^a, p^l) and policy (τ, T), a household of type π chooses annuity holdings a, life-insurance holdings l, saving s, and consumption-bequest allocations (c_1, c_2, b). The household's problem is

$$\max_{c_1,c_2,b,a,l,s} u(c_1) + \pi\beta U(c_2) + (1-\pi)\beta v(b) \tag{1}$$

subject to

$$\begin{aligned} c_1 + p^a a + p^l l + s &\leq e(1-\tau) \\ c_2 &\leq As + a + T \\ b &\leq As + l \\ c_1, c_2, b, a, l, s &\geq 0 \end{aligned}$$

where T is the Social Security transfer.

Following Bisin and Gottardi (2003) and Dubey and Geanakoplos (2002), intermediaries take expected per-capita contributions and payouts as given and choose pool size.[3] Let R^a and Q^a denote per-capita contributions and payouts in the annuity pool, and let R^l and Q^l denote the corresponding objects for life insurance. The intermediary chooses the fraction of the population that can join each pool, n^a and n^l.

[2] Ales and Maziero (2011) and Attar et al. (2014) show that almost linear contracts can emerge under nonexclusive trade in static settings. Cawley and Philipson (1999) provide evidence of approximately linear prices in life insurance, and Cannon and Tonks (2008) and Finkelstein and Poterba (2004) discuss linear pricing in annuity markets.

[3] Relative to Dubey and Geanakoplos (2002), pools here are identical and participation is nonexclusive.

$$\max_{0 \le n^a, n^l \le 1, k \in \mathbb{R}_+} n^a R^a + n^l R^l - k \tag{2}$$

subject to

$$n^a Q^a + n^l Q^l \le Ak$$

Since intermediaries are identical and the technology is constant returns to scale, there is no loss in treating them as a representative competitive fringe. In equilibrium, expected contributions and payouts must be consistent with household behavior and market prices.

Definition 1 *A Competitive Equilibrium with Asymmetric Information consists of household allocations*

$$\{c_1^*(\pi), c_2^*(\pi), b^*(\pi), a^*(\pi), l^*(\pi), s^*(\pi)\}_{\pi \in [\underline{\pi}, \bar{\pi}]},$$

intermediary choices $\{(n^{j*})_{j=a,l}, k^*\}$*, contract prices* (p^{a*}, p^{l*})*, anticipated contributions and payouts in each pool* $\{(R^{j*}, Q^{j*})_{j=a,l}\}$*, and Social Security policy* (τ, T) *such that*

1. *Household optimization: for each* $\pi \in [\underline{\pi}, \bar{\pi}]$*,*

$$\{c_1^*(\pi), c_2^*(\pi), b^*(\pi), a^*(\pi), l^*(\pi), s^*(\pi)\}$$

   solves (1) at prices (p^{a*}, p^{l*}) *and policy* (τ, T)*.*

2. *Intermediary optimization: given* $(R^{j*}, Q^{j*})_{j=a,l}$*, the choices* $\{(n^{j*})_{j=a,l}, k^*\}$ *solve (2).*

3. $\{(R^{j*}, Q^{j*})_{j=a,l}\}$ *satisfy the following consistency conditions:*

$$\begin{aligned} R^{a*} &= \int_{\underline{\pi}}^{\bar{\pi}} p^{a*} a^*(\pi) d\mu(\pi) \\ R^{l*} &= \int_{\underline{\pi}}^{\bar{\pi}} p^{l*} l^*(\pi) d\mu(\pi) \end{aligned} \tag{3}$$

   and

$$\begin{aligned} Q^{a*} &= \int_{\underline{\pi}}^{\bar{\pi}} \pi a^*(\pi) d\mu(\pi) \\ Q^{l*} &= \int_{\underline{\pi}}^{\bar{\pi}} (1-\pi) l^*(\pi) d\mu(\pi) \end{aligned} \tag{4}$$

4. *Social Security budget:*

$$A\tau e = T\mathbf{E}[\pi]$$

   (where $\mathbf{E}[\pi] = \int_{\underline{\pi}}^{\bar{\pi}} \pi d\mu(\pi)$ *)*

5. *Contract markets clear:*

$$n^{j*} = 1 \quad j = a, l$$

6. *Goods market clears:*

$$\int_{\underline{\pi}}^{\bar{\pi}} [\pi c_2^*(\pi) + (1-\pi) b^*(\pi)] d\mu(\pi) = A \left(e - \int_{\underline{\pi}}^{\bar{\pi}} c_1^*(\pi) d\mu(\pi) \right)$$

3 Equilibrium Properties and Existence

3.1 Full Information Economy

This section derives the paper's main equilibrium results. I begin with a full-information benchmark, where each household faces actuarially fair annuity and life-insurance prices. The benchmark isolates the sign of net annuity demand. I then turn to the asymmetric-information economy, characterize individual demand, prove existence, and show that at most one private insurance market is active.

Under full information, type is public and insurance pools are type-specific. Each consumer of type π therefore faces fair prices, $p^a(\pi) = \frac{\pi}{A}$ and $p^l(\pi) = \frac{1-\pi}{A}$. The consumer's problem is

$$\max_{c_1, c_2, b, a, l, s} u(c_1) + \pi \beta U(c_2) + (1-\pi)\beta v(b)$$

subject to

$$\begin{aligned} c_1 + \frac{\pi}{A} a + \frac{1-\pi}{A} l + s &\leq e(1-\tau) \\ c_2 &\leq As + a + T \\ b &\leq As + l \\ c_1, c_2, b, a, l, s &\geq 0 \end{aligned}$$

It is a straight forward application of maximum theorem to show that solutions of the above problem are continuous functions of π. Solution to this problem is characterized by the following first order conditions

$$\frac{u'(c_1(\pi))}{A\beta} = U'(c_2(\pi)) = v'(b(\pi)) \tag{5}$$

if $a(\pi) > 0$ and $l(\pi) > 0$ and

$$\frac{u'(c_1(\pi))}{A\beta} > U'(c_2(\pi)) \text{ and } \frac{u'(c_1(\pi))}{A\beta} = v'(b) \tag{6}$$

if $a(\pi) = s(\pi) = 0$.[4]

In both cases the sign of $a(\pi) - l(\pi)$, which I call *net annuity purchase*, has a special property

Proposition 1 *In any full-information equilibrium, net annuity purchase has the same sign for all types. That is, if $a(\pi) - l(\pi)$ is positive, negative, or zero for one type, it has the same sign for every $\pi \in [\underline{\pi}, \bar{\pi}]$.*

Proof sketch. If one type chooses zero net annuity, the resulting allocation can be replicated by every type because the full-information budget equations no longer depend on survival probability. Continuity then rules out an equilibrium in which net annuity demand changes sign across types. The full proof is in the Appendix proof of Proposition 1.

This proposition gives the key full-information benchmark: net annuity demand has the same sign for all types. The next proposition shows that Social Security shifts this sign in a simple way. There is a critical policy level at which households are exactly indifferent between annuity and life-insurance positions. Below it, net annuity demand is positive for all types. Above it, net annuity demand is negative for all types.

Proposition 2 *In the full-information economy there exists a Social Security tax τ^* such that all types choose positive net annuity for $\tau < \tau^*$, all types choose negative net annuity for $\tau > \tau^*$, and all types choose zero net annuity at $\tau = \tau^*$.*

Proof sketch. Construct the policy that finances the allocation associated with the average survival probability. At that policy, net annuity demand is exactly zero. Moving the tax below or above this point lowers or raises public annuitization, so the average type switches from positive to negative net annuity demand. The previous proposition then extends the sign to all types. The full proof is in the Appendix proof of Proposition 2.

The same cutoff policy will reappear in the asymmetric-information economy. It separates an annuity region from a life-insurance region and later serves as the policy level that implements the efficient allocation.

[4] Because of INADA condition on $v(\cdot)$, s and l cannot be both equal to zero.

3.2 Asymmetric Information Economy

I now turn to the competitive economy with private information. The first task is to characterize equilibrium prices and household demand. The second is to show that these objects fit together in a fixed point. When intermediaries have correct expectations about contributions and payouts, equilibrium prices satisfy

$$Ap^a \int_{\underline{\pi}}^{\bar{\pi}} a(\pi) d\mu(\pi) = \int_{\underline{\pi}}^{\bar{\pi}} \pi a(\pi) d\mu(\pi) \tag{7}$$

and

$$Ap^l \int_{\underline{\pi}}^{\bar{\pi}} l(\pi) d\mu(\pi) = \int_{\underline{\pi}}^{\bar{\pi}} (1-\pi) l(\pi) d\mu(\pi) \tag{8}$$

Note however that when $\int_{\underline{\pi}}^{\bar{\pi}} a(\pi)d\mu(\pi) = 0$ or $\int_{\underline{\pi}}^{\bar{\pi}} l(\pi)d\mu(\pi) = 0$ the price is indeterminate by these equations. In other words any price will satisfy these equation when aggregate demand for insurance is zero (since we get zero on both sides of the equations). For now, in such cases I choose the prices to be the following

$$p^a = \frac{\bar{\pi}}{A} \text{ whenever } \int_{\underline{\pi}}^{\bar{\pi}} a(\pi) d\mu(\pi) = 0 \tag{9}$$

and

$$p^l = \frac{1-\underline{\pi}}{A} \text{ whenever } \int_{\underline{\pi}}^{\bar{\pi}} l(\pi) d\mu(\pi) = 0$$

Later (in the proof of existence) I will justify this selection and show that they are in fact consistent with equilibrium behavior of consumers.

The next lemma is a sorting result that will be used repeatedly. It shows that when higher-risk households buy more insurance, the equilibrium price of that contract exceeds the price implied by average population risk.

Lemma 1 *Consider a function f on $[\underline{\pi}, \bar{\pi}]$ such that $f(\pi) \lesseqqgtr 0$ as $\pi \lesseqqgtr \pi^*$ and $\int_{\underline{\pi}}^{\bar{\pi}} f(\pi)d\mu(\pi) = 0$. Let $g(\pi)$ be an increasing function with $g(\bar{\pi}) > g(\underline{\pi})$. Then $\int_{\underline{\pi}}^{\bar{\pi}} f(\pi)g(\pi)d\mu(\pi) > 0$*

Proof. Since f changes sign once and sums to zero, applying the larger weights from g to the upper end of the type distribution tilts the integral toward the positive part. The full derivation is in the Appendix proof of Lemma 1.

I now characterize consumer behavior. The first step is to show that equilibrium prices must satisfy $A(p^a + p^l) > 1$. This inequality implies that a household never wants to hold both annuity and life-insurance positions at the same time.

Lemma 2 *Any equilibrium price vector satisfies $A(p^a + p^l) \geq 1$.*

Proof sketch. If $A(p^a + p^l) < 1$, a household can dominate saving by combining annuity and life-insurance positions. Once types sort into the two pools, annuity buyers are selected toward high survival types and life-insurance buyers toward low survival types, so both prices must rise above their population-average values. That contradicts $A(p^a + p^l) < 1$. The full proof is in the Appendix proof of Lemma 2.

The above lemma restrict equilibrium prices to be $A(p^a + p^l) \geq 1$. The next lemma shows that the equality ($A(p^a + p^l) = 1$) is also incompatible with any equilibrium.

Lemma 3 *No equilibrium can satisfy $A(p^a + p^l) = 1$.*

Proof sketch. At equality, households are exactly indifferent at the margin between annuity, life-insurance, and saving positions. But the same sorting argument still implies net annuity demand rises with survival probability. The pricing equations then force both contracts to trade at prices above their population-average values, which is incompatible with equality. The full proof is in the Appendix proof of Lemma 3.

So far I have established that any equilibrium price must satisfy the following inequality

$$A(p^a + p^l) > 1 \tag{10}$$

Next, I will focus on characterizing the consumer behavior under the admissible prices that satisfy inequality (10). An important implication of this property is that purchasing both annuity and life insurance is more expensive than saving. Therefore, each consumer prefers to purchase at most one of them (and possibly hold some savings). This is formally shown in the next lemma.

Lemma 4 *For any prices with $A(p^a + p^l) > 1$ (and not just the the equilibrium prices) and any tax and transfer (τ, T)*

1. *Consumer choices, $\{c_1(\pi), c_2(\pi), b(\pi), a(\pi), l(\pi), s(\pi)\}$ are continuous in π*
2. *$a(\pi)$ is weakly increasing in π and $l(\pi)$ is weakly decreasing in π*
3. *There are cutoff survival probabilities $\underline{\pi}_a$ and $\overline{\pi}_l$ such that $\underline{\pi}_a > \overline{\pi}_l$ and*

if	$\pi < \overline{\pi}_l$	*consumer purchases life insurance and not annuity*
if	$\pi \in [\overline{\pi}_l, \underline{\pi}_a]$	*consumer is not insured*
if	$\pi > \underline{\pi}_a$	*consumer purchases annuity and not life insurance*

(Note that this allows for the possibility that $\overline{\pi}_l < \underline{\pi}$ or $\underline{\pi}_a > \overline{\pi}$ or both.)

Proof sketch. Continuity comes from the household problem. The price inequality makes it too costly to combine annuities and life insurance, so each type chooses at most one of them. Since higher survival types value annuities more and lower survival types value life insurance more, demand sorts monotonically and yields cutoff types separating the three regions. The full proof is in the Appendix proof of Lemma 4.

The previous lemma delivers the basic demand structure. At any admissible price vector, households sort into three groups: life-insurance buyers, self-insurers, and annuity buyers. The cutoffs $\underline{\pi}_a$ and $\overline{\pi}_l$ determine both participation and the composition of risk in each pool. This is the channel through which selection enters equilibrium prices.

The cutoffs are the types indifferent between insurance and self-insurance. Because individual demand is continuous, the cutoffs are continuous in prices. This continuity makes it possible to formulate the equilibrium problem as a fixed point in prices. The next proposition establishes existence.

Proposition 3 *A competitive equilibrium exists.*

Proof sketch. Use the cutoff structure to define price-updating functions for annuities and life insurance. Short-sale restrictions keep these maps well behaved even when one market is inactive. Because they are continuous and map a compact set into itself, Brouwer's fixed-point theorem yields equilibrium prices. The full proof is in the Appendix proof of Proposition 3.

The short-sale restrictions on annuity and life-insurance holdings are important for this existence result. If households could both buy and short the same contract, a single linear price would generally not be sufficient to clear the market under adverse selection. The restriction separates households from intermediaries and keeps the pooling problem one-dimensional.[5]

As Bisin and Gottardi (1999) emphasize, the underlying difficulty is that under asymmetric information the same contract is a different commodity for different types. Households value a contract differently, and their choices affect its return. Aggregate returns are therefore not linear in aggregate positions. A short-sale restriction separates buyers from sellers and makes linear pricing feasible in the present environment.[6]

[5] If some types held long positions and others held short positions in the same annuity pool, aggregate contributions could be zero while expected payouts remained strictly positive.

[6] Bisin and Gottardi (1999) propose an alternative separation device based on nonlinear bid-ask pricing.

The existence result also contrasts with the nonexistence literature following Rothschild and Stiglitz (1976). In that literature, insurers offer menus of exclusive contracts and can profitably attract particular types outside the candidate equilibrium set. Here I rule out that form of entry by disallowing monitoring and by treating pools as identical.[7]

Existence does not imply that both private insurance pools are active. An insurance pool is active if aggregate contributions to that pool are positive. The next theorem is the main result of the paper: at most one private insurance market is active in equilibrium.

The intuition is simple. Under full information, all types choose net annuity positions of the same sign. The asymmetric-information economy inherits this property at the types that face fair prices in active markets. If both annuity and life-insurance markets were active, one type would face a fair annuity price and choose positive net annuity, while another type would face a fair life-insurance price and choose negative net annuity. The full-information benchmark rules this out.

Theorem 1 *In any equilibrium, at most one private insurance market is active.*

Proof sketch. If the annuity market is active, some type must face a fair annuity price and therefore choose positive net annuity. If the life-insurance market is also active, another type must face a fair life-insurance price and therefore choose negative net annuity. The full-information benchmark rules out such a sign reversal across types. The full proof is in the Appendix proof of Theorem 1.

4 Policy

4.1 Ex Ante Efficient Allocations

This section records two consequences of the market structure characterized above. The policy instrument is public annuitization through Social Security. I first characterize the ex ante efficient allocation and show how the cutoff policy from the full-information benchmark reappears in the private-information economy. I then study how Social Security changes private-market prices.

Before discussing policy, it is useful to define the efficient allocation.

[7] Dubey and Geanakoplos (2002) allow finitely many pools with different capacities and recover similar instability.

Definition 2 *An allocation is Ex-ante Pareto Efficient if it is the solution to the following problem*

$$\max \mathbf{E}_\pi \left[u_1(c_1(\pi)) + \pi U_2(c_2(\pi)) + (1-\pi)v(b(\pi)) \right]$$

subject to

$$\mathbf{E}_\pi \left[c_1(\pi) + \frac{1}{A}\pi c_2(\pi) + \frac{1}{A}(1-\pi)b(\pi) \right] = e$$

First order conditions imply that efficient allocations must satisfy

$$u'(c_1(\pi)) = A\beta U'(c_2(\pi)) = A\beta v'(b(\pi))$$

and feasibility

$$\mathbf{E}_\pi \left[c_1(\pi) + \frac{1}{A}\pi c_2(\pi) + \frac{1}{A}(1-\pi)b(\pi) \right] = e$$

Note that this immediately implies that efficient allocations are constant across types. In this environment, private information does not distort the planner's allocation directly; it distorts the allocation through market prices and participation. This makes the efficient allocation easy to implement with a tax-transfer policy. The next theorem records this implication of the model.

Theorem 2 *There exists a Social Security tax τ^* such that all consumers purchase annuities for $\tau < \tau^*$, all consumers purchase life insurance for $\tau > \tau^*$, and no consumer purchases private insurance at $\tau = \tau^*$. This cutoff is the same τ^* as in the full-information economy.*

Proof sketch. At the cutoff policy, the efficient allocation is feasible for every type and leaves no role for private insurance. Away from the cutoff, public annuitization is too low or too high relative to that benchmark, so some type strictly prefers annuities below the cutoff and life insurance above it. The full proof is in the Appendix proof of Theorem 2.

The economic interpretation is straightforward. Social Security is a substitute for private annuity income, so higher benefits reduce private annuity demand directly. They also change pool composition: lower-survival households exit the annuity market first, raising the price faced by those who remain. At the same time, better insurance in the survival state increases the relative attractiveness of life insurance. The next proposition states the associated implementation result.

Proposition 4 *The Social Security policy (τ^*, T^*) implements the ex-ante efficient allocation.*

Proof. This follows directly from the previous theorem. A full proof is recorded in the Appendix proof of Proposition 4 for completeness.

4.2 Effect of Social Security on Prices

I now turn from participation to prices. In the annuity region, higher Social Security benefits reduce annuity demand for all types, but the reduction is strongest for lower-survival households. This worsens selection in the annuity pool and raises the annuity price. In the life-insurance region the same force works in the opposite direction. The next proposition establishes these comparative statics under homothetic preferences.

Proposition 5 *Suppose preferences are homothetic: let u be homothetic, $U = \alpha u$ and $v = \gamma u$ for some constants α and γ. Let p^{a*} and p^{l*} be equilibrium annuity and life insurance prices. Then*

$$\frac{\partial p^{a*}}{\partial \tau} > 0 \text{ and } \frac{\partial p^{l*}}{\partial \tau} = 0 \text{ for } \tau < \tau^*$$

$$\frac{\partial p^{l*}}{\partial \tau} < 0 \text{ and } \frac{\partial p^{a*}}{\partial \tau} = 0 \text{ for } \tau > \tau^*$$

Proof sketch. Under homothetic preferences, insurance demand scales with disposable resources. Higher Social Security reduces annuity demand for all annuity buyers, but especially for the lower-survival types at the bottom of that pool, worsening selection and raising the annuity price. Above the cutoff the same logic applies to life insurance with the sign reversed. The full proof is in the Appendix proof of Proposition 5.

4.3 Observable Implications

The model yields a small set of predictions that connect the equilibrium results to observed insurance-market outcomes. First, greater public annuitization reduces private annuity participation and raises annuity prices in the region where annuities remain active. Second, once public annuitization is high enough, the active private market shifts from annuities to life insurance. Third, market composition matters as much as aggregate quantity: the price of a private mortality-contingent contract depends on which types remain in the pool. These implications give the model an interpretation of the observed asymmetry between thin private annuity markets and much larger life-insurance markets.

4.4 Numerical Illustration

To illustrate the mechanism, I calibrate a two-period version of the model using the homothetic structure of Proposition 5. The exercise is a numerical illustration of the equilibrium objects emphasized above: active markets, participation cutoffs, and pooling prices. The calibration uses a gross return of $A = 1.03$, a discount factor of $\beta = 0.97$, and log preferences with $U(c) = \log c$ and $v(b) = \gamma \log b$. I set $\gamma = 0.12$. Survival probabilities lie on a 100-point grid over $[0.60, 0.99]$. Their cross-sectional distribution is taken to be beta, with parameters chosen to match an average survival probability of 0.80 and a standard deviation of 0.09. These moments are chosen to produce a dispersed mortality distribution in the spirit of Hosseini (2015). The bequest motive and the interpretation of low annuitization are guided by Pashchenko (2013). For each value of τ, I begin from a price guess, solve household annuity and life-insurance demands from the first-order conditions and budget constraints, update the pooling price from the zero-profit condition, and iterate to convergence. The exercise remains fully within the paper's two-period environment.

Table 1: Baseline Numerical Illustration

τ	Active market	Price	Buyer share	Cutoff type
0.00	Annuities	0.784	0.996	$\underline{\pi}_a = 0.616$
0.05	Annuities	0.799	0.899	$\underline{\pi}_a = 0.679$
0.25	None	–	0.000	–
0.40	Life insurance	0.356	0.083	$\overline{\pi}_l = 0.667$

The numerical exercise reproduces the qualitative structure of the theory. At low values of τ, only the annuity market is active. As public annuitization rises, the annuity-buying pool shrinks from the bottom: lower-survival households leave first, the annuity cutoff rises, and the equilibrium annuity price increases. Over an intermediate range of τ there is no private insurance trade. For higher values of τ, the active private market switches to life insurance, and the life-insurance pool expands as public annuitization increases further. In this calibration the switch occurs only for relatively high values of τ, beginning around $\tau = 0.40$. Figure 1 plots the share of private annuity and life-insurance buyers as Social Security changes. Figure 2 plots the associated equilibrium prices. The point of the exercise is to show that the market-switching result and the associated price effects arise for plausible parameter values in a simple calibration.

5 Discussion

The model is deliberately simple. It has two periods, linear insurance contracts, no aggregate uncertainty, nonexclusive trade, and short-sale restrictions on insurance holdings. These assumptions make it possible to characterize equilibrium prices, participation, and market structure analytically. They also shape the sharpness of the results.

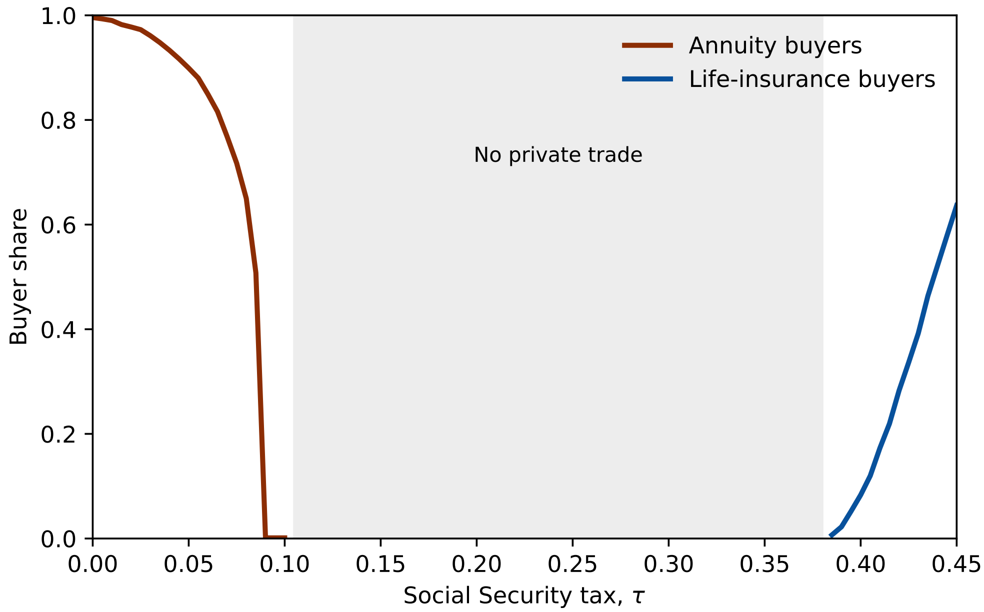

Figure 1: Private Insurance Buyer Shares

The result that at most one private insurance market is active should be read in this way. The theorem identifies a force that pushes annuities and life insurance apart in equilibrium when mortality risk is private information and households have access to both contracts. In the data, private annuities are not literally absent, and life insurance does not exhaust private insurance activity. The model abstracts from many features that would soften the prediction, including richer contract menus, underwriting, household heterogeneity beyond survival risk, and dynamic trading opportunities.

The policy result should be read in the same spirit. In the present environment, efficient allocations are constant across types, so Social Security can implement them with a simple tax-transfer rule. That result is useful because it clarifies the interaction between public annuitization, selection, and private-market activity. It does not imply that optimal policy would take the same form in richer environments where contracts are nonlinear, insurer strategies are endogenous, or efficient allocations vary across observable or unobservable household characteristics.

These considerations suggest two natural extensions. One is to study more general contract spaces and insurer competition. The other is to embed the mechanism analyzed here in a quantitative environment, so that the selection effect on private annuity and life-insurance markets can be compared with the magnitudes observed in the data.

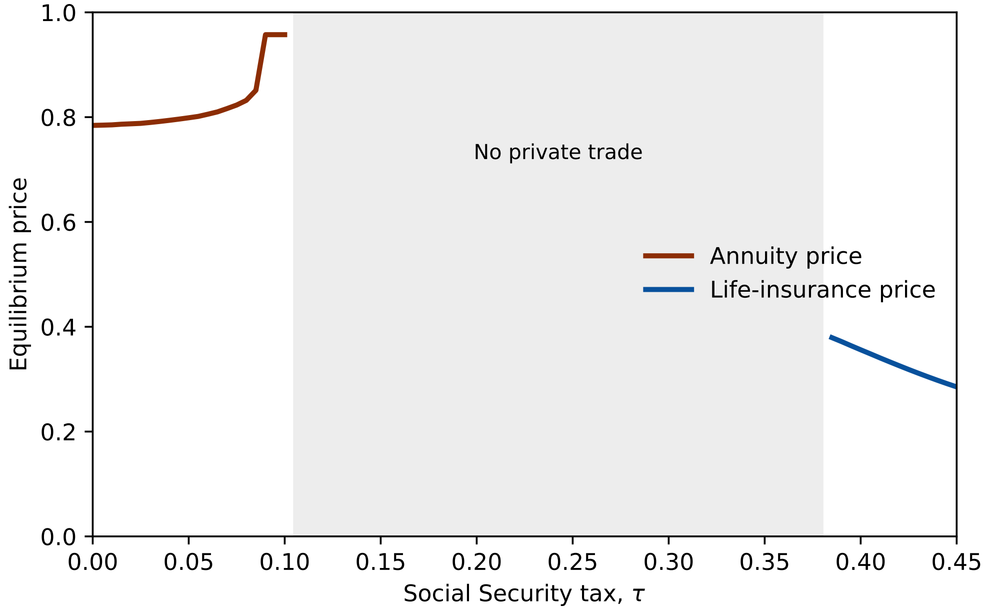

FIGURE 2: Equilibrium Prices and Public Annuitization

6 Concluding Remarks

This paper studies a competitive model of annuity and life insurance with private information about survival probabilities. Households differ in mortality risk, value both retirement consumption and bequests, and trade insurance through competitive pools. Social Security supplies public annuitization. In this environment equilibrium exists and has a simple structure: at most one private insurance market is active.

This structure gives the model its main implication. Public annuitization changes not only the volume of private annuity trade but also the composition of insurance pools and the identity of the active private market. Higher Social Security benefits reduce annuity demand, increase annuity prices, and shift the economy toward life insurance. These are equilibrium effects of selection.

The model also yields an ex ante efficient allocation that can be implemented by a tax-transfer policy. In the implementing allocation private insurance trade disappears. This result follows from the fact that efficient allocations are constant across types in the present environment.

The analysis is based on linear contracts, nonexclusive trade, and short-sale restrictions on insurance holdings. These assumptions produce a tractable competitive theory of joint annuity and life insurance markets under adverse selection. The numerical illustration shows that the market-switching mechanism appears in a simple calibration and generates monotone changes in participation and prices as public annuitization rises. Extending the analysis to richer contract spaces and more explicit insurer strategies would be a useful next step.

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Proofs

Proof of Proposition 1

First I show that if one type chooses $a(\pi) = l(\pi)$, then all types must do so. Suppose type π chooses $a(\pi) = l(\pi)$. Then the allocation that he chooses must satisfy first order conditions (5) or (6) together with budget constraint, which after imposing $a(\pi) = l(\pi)$ becomes

$$\begin{aligned} c_1(\pi) + \frac{1}{A}(l(\pi) + As(\pi)) &\leq e(1-\tau) \\ c_2(\pi) &\leq Tr + l(\pi) + As(\pi) \\ b(\pi) &\leq l(\pi) + As(\pi) \end{aligned}$$

Note that these equations do not depend on π. Therefore, they must hold for all types $\pi \in [\underline{\pi}, \bar{\pi}]$.

Now suppose for some type $a(\pi) - l(\pi) > 0$ and there is some other type such that $a(\pi') - l(\pi') < 0$. Then by continuity there must exist a $\tilde{\pi} \in [\underline{\pi}, \bar{\pi}]$ such that $a(\tilde{\pi}) - l(\tilde{\pi}) = 0$. This is a contradiction.

Proof of Proposition 2

Consider the allocation that satisfies the following conditions.

$$u'(c_1^*) = U'(c_2^*) = v'(b^*)$$

$$c_1^* + \frac{1}{A}\mathbf{E}[\pi]c_2^* + \frac{1}{A}(1 - \mathbf{E}[\pi])b^* = e$$

where $\mathbf{E}[\pi]$ is average survival probability in the economy. Let τ^* be such that

$$c_1^* + \frac{1}{A}b^* = e(1 - \tau^*)$$

and therefore, $T^* = c_2^* - b^* = \frac{e\tau^*}{\mathbf{E}[\pi]}$. These allocations can only be achieved by choosing $b^* = l^* + As^*$ and $l^* = a^*$. Furthermore, they are independent of type and hence are the same for every type $\pi \in [\underline{\pi}, \bar{\pi}]$.

The claim is that for tax and transfer (τ^*, T^*) no consumer purchases any private insurance and all types choose the same allocation, the ex-ante efficient allocation. This follows because the starred allocation is independent of type and is the unique allocation that satisfies the first order conditions and budget constraint of every type.

Now consider the choice of the type $\pi' = \mathbf{E}[\pi]$. This consumer's consumption and bequest are independent of τ and T, and always equal the starred allocation defined above. Consider this type's budget constraint for $T < T^*$

$$a(\pi') - l(\pi') = c_2^* - b^* - T > c_1^* - b^* - T^* = 0$$

Since one type purchases positive net annuity, all the types must do so. Now consider the case where $T > T^*$. The budget constraint of type $\pi' = \mathbf{E}[\pi]$ is

$$\begin{aligned} c_1^* + \frac{1}{A}\pi' a(\pi') + \frac{1}{A}(1 - \pi')l(\pi') + s(\pi') &= e(1 - \tau) \\ c_2^* &= T + a(\pi') + As(\pi') \\ b^* &= As(\pi') + l(\pi') \end{aligned}$$

Substituting for $s(\pi')$ in the second equation gives

$$a(\pi') - l(\pi') = c_2^* - b^* - T < c_1^* - b^* - T^* = 0$$

Proof of Lemma 1

$$\begin{aligned}\int_{\underline{\pi}}^{\bar{\pi}} f(\pi)g(\pi)d\mu(\pi) &= \int_{\underline{\pi}}^{\pi^*} f(\pi)g(\pi)d\mu(\pi) + \int_{\pi^*}^{\bar{\pi}} f(\pi)g(\pi)d\mu(\pi) \\ &> g(\pi^*)\left[\int_{\underline{\pi}}^{\bar{\pi}} f(\pi)d\mu(\pi)\right] = 0\end{aligned}$$

Proof of Lemma 2

Suppose $A(p^a + p^l) < 1$ and consider the first order conditions of the consumer.

$$u'(c_1) \geq \pi A\beta U'(c_2) + (1-\pi)\beta A v'(b) \tag{11}$$

$$p^a u'(c_1) \geq \pi\beta U'(c_2) \tag{12}$$

$$p^l u'(c_1) \geq (1-\pi)\beta v'(b) \tag{13}$$

Under these prices $s(\pi)$ cannot be positive for any type. This follows from the no-arbitrage condition: setting (11) at equality and adding (12) and (13) yields a contradiction. Therefore we must have $s(\pi) = 0$. Then Assumption 2 implies $l(\pi) = b(\pi) > 0$ for all π. $a(\pi)$ can be either positive of zero. I first show that $l(\pi)$ is strictly decreasing function of π in both cases. Suppose $a(\pi) = 0$, then solution to consumers problem is characterized by

$$p^l u'(c_1(\pi)) = \beta(1-\pi)v'(b(\pi))$$

$$c_1(\pi) + p^l b(\pi) = e(1-\tau) \text{ and } c_2(\pi) = T$$

Now consider problem of a type $\pi' > \pi$. Consider the first order condition for this type at the $c(\pi)$ and $l(\pi)$ allocations.

$$p^l u'(c_1(\pi)) > \beta(1-\pi')v'(b(\pi))$$

Suppose $a(\pi') = 0$. It is clear that both $b(\pi)$ and $c_1(\pi)$ cannot increase or decrease at the same time. Then the only way to restore equality is to have $c_1(\pi') > c_1(\pi)$ and $b(\pi') < b(\pi)$. If $a(\pi') > 0$, then $c_2(\pi') > c_2(\pi)$. Also, c_1 and b cannot increase at the same time (since $c_1 + p^l b$ must decrease). The only possibility that doesn't have $b(\pi') < b(\pi)$ is the case where $b(\pi') > b(\pi)$ and $c(\pi') < c(\pi)$. But this cannot restore equality in first order condition (which is required for $b(\pi') > 0$). Therefore, the claim is established that $b(\pi)$ is a strictly decreasing function of π. Now consider the price equation (8). We can rewrite it as (since $l(\pi) = b(\pi) > 0$ for all π)

$$Ap^l = \frac{\int_{\underline{\pi}}^{\bar{\pi}} (1-\pi) l(\pi) d\mu(\pi)}{\int_{\underline{\pi}}^{\bar{\pi}} l(\pi) d\mu(\pi)}$$

Using the result of the previous lemma we can show that $Ap^l > (1 - \mathbf{E}[\pi])$. Now if $a(\pi) = 0$ for all π, we know that $Ap^a = \bar{\pi}$. If $a(\pi) > 0$ for some π using similar argument as above we can show that it must be a strictly increasing function of π. The argument is as follows. Consider the choices of type π

$$\begin{aligned} p^l u'(c_1(\pi)) &= \beta(1-\pi) v'(b(\pi)) \\ p^a u'(c_1(\pi)) &= \beta \pi U'(c_2(\pi)) \end{aligned}$$

$$c_1(\pi) + p^l b(\pi) + p^a a(\pi) = e(1-\tau) \text{ and } c_2(\pi) = T + a(\pi)$$

Now suppose $\pi' > \pi$. Evaluate the first order conditions of type π' at $c_1(\pi)$, $c_2(\pi)$ and $b(\pi)$

$$\begin{aligned} p^l u'(c_1(\pi)) &> \beta(1-\pi') v'(b(\pi)) \\ p^a u'(c_1(\pi)) &< \beta \pi' U'(c_2(\pi)) \end{aligned}$$

and note that $c_1(\pi) + p^l b(\pi) + p^a a(\pi) = e(1-\tau)$. The only possible way to restore the equality without violating budget constraint is to have $c_2(\pi') > c_2(\pi)$ (and therefore $a(\pi') > a(\pi)$), $c_1(\pi') < c_1(\pi)$ and $b(\pi') < b(\pi)$. Again, applying the previous lemma to equation (7) we can show that $Ap^a > \mathbf{E}[\pi]$. In both cases we get $A(p^a + p^l) > 1$. A contradiction.

Proof of Lemma 3

Again consider the consumer problem, substituting for $a(\pi)$ and $l(\pi)$ in the budget constraint:

$$\begin{aligned} p^l u'(c_1(\pi)) &= \beta(1-\pi) v'(b(\pi)) \\ p^a u'(c_1(\pi)) &= \beta \pi U'(c_2(\pi)) \end{aligned}$$

$$c_1(\pi) + p^a c_1(\pi) + p^l b(\pi) = e(1-\tau) + p^a T$$

Note also that when prices are such that $A(p^a + p^l) = 1$ the first order conditions hold with equality and portfolio composition of consumer is indeterminate (consumer is indifferent between any combination of a, l and s). First I show that $c_2(\pi)$ (respectively $b(\pi)$) is strictly increasing (respectively decreasing) in π. The argument is the same as in the previous lemma. Consider the first order condition of type π

$$\begin{aligned} p^l u'(c_1(\pi)) &= \beta(1-\pi)v'(b(\pi)) \\ p^a u'(c_1(\pi)) &= \beta\pi U'(c_2(\pi)) \end{aligned}$$

now suppose $\pi' > \pi$ and evaluate this types first order condition at $c_1(\pi)$, $c_2(\pi)$ and $b(\pi)$

$$\begin{aligned} p^l u'(c_1(\pi)) &> \beta(1-\pi')v'(b(\pi)) \\ p^a u'(c_1(\pi)) &< \beta\pi' U'(c_2(\pi)) \end{aligned}$$

then the only way to restore equality without violating budget constraint is to have $c_1(\pi') < c_1(\pi)$, $c_2(\pi') > c_2(\pi)$ and $b(\pi') < b(\pi)$. This also implies that net annuity purchase, $a(\pi) - l(\pi) = c_2(\pi) - b(\pi) - T$, is strictly increasing.

Next step is to show that $\int_{\underline{\pi}}^{\bar{\pi}}(a(\pi) - l(\pi))d\mu(\pi) \neq 0$. Suppose otherwise. Since $a(\pi) - l(\pi)$ is monotone and continuous, there must exist a π^* such that $a(\pi^*) - l(\pi^*) = 0$. Now recall pricing equations (7) and (8) and replace $Ap^l = 1 - Ap^a$

$$Ap^a \int_{\underline{\pi}}^{\bar{\pi}} a(\pi)d\mu(\pi) = \int_{\underline{\pi}}^{\bar{\pi}} \pi a(\pi)d\mu(\pi)$$

$$(1 - Ap^a) \int_{\underline{\pi}}^{\bar{\pi}} l(\pi)d\mu(\pi) = \int_{\underline{\pi}}^{\bar{\pi}} (1-\pi)l(\pi)d\mu(\pi)$$

Adding the above equations gives

$$Ap^a \int_{\underline{\pi}}^{\bar{\pi}} (a(\pi) - l(\pi))d\mu(\pi) = \int_{\underline{\pi}}^{\bar{\pi}} \pi(a(\pi) - l(\pi))d\mu(\pi) > 0$$

where the last inequality follows from the previous lemma. This is a contradiction. Therefore, $\int_{\underline{\pi}}^{\bar{\pi}}(a(\pi) - l(\pi))d\mu(\pi) \neq 0$. Now consider the same pricing equations as above

$$Ap^a = \frac{\int_{\underline{\pi}}^{\bar{\pi}} \pi(a(\pi) - l(\pi))d\mu(\pi)}{\int_{\underline{\pi}}^{\bar{\pi}}(a(\pi) - l(\pi))d\mu(\pi)} > \mathbf{E}[\pi]$$

Similarly we can show that $Ap^l > 1 - \mathbf{E}[\pi]$. This is a contradiction with $A(p^a + p^l) = 1$.

Proof of Lemma 4

Proof of 1) The first part is a direct application of the Maximum Theorem.

Proof of 2) For the second part first note that the assumption on prices that holdings of annuity and life insurance cannot be positive at the same time: adding the first order conditions for a and l and comparing the result with the first order condition for savings yields a contradiction. So each consumer either buys annuity or life insurance or none of them. Suppose that $a(\pi) > 0$ and consider $a(\pi')$ where $\pi' < \pi$. Note that as I just argued, this implies $l(\pi) = 0$. Then either $a(\pi') = 0$, in which case the proof is done, or $a(\pi') > 0$ (and hence $l(\pi') = 0$). When $l(\pi)$ and $(l(\pi'))$ are zero, by the INADA condition $s(\pi) = b(\pi)$ (and $s(\pi') = b(\pi'))$ has to be positive. First order conditions for π are

$$\begin{aligned} u'(c_1(\pi)) &= \pi A\beta U'(c_2(\pi)) + (1-\pi)\beta A v'(b(\pi)) \\ p^a u'(c_1(\pi)) &= \pi\beta U'(c_2(\pi)) \\ p^l u'(c_1(\pi)) &> (1-\pi)\beta v'(b(\pi)) \end{aligned}$$

From top two equations we get

$$\begin{aligned} p^a u'(c_1(\pi)) &= \beta\pi U'(c_2(\pi)) \\ (1 - Ap^a)u'(c_1(\pi)) &= (1-\pi)v'(b(\pi)) \end{aligned}$$

Also the budget constraint must be satisfied for all types (hence I suppress the π argument)

$$c_1 + p^a c_2 + (1 - Ap^a)b = e(1-\tau) + p^a T$$

Now consider type $\pi' < \pi$ and evaluate its first order conditions at the allocations chosen by type π:

$$\begin{aligned} p^a u'(c_1) &> \beta\pi' U'(c_2) \\ (1 - Ap^a)u'(c_1) &< (1-\pi')v'(b) \end{aligned}$$

The only allocations that restore equality in these equations and remain consistent with the budget constraint must satisfy

$$c_1(\pi') > c_1(\pi), c_2(\pi') < c_2(\pi), b(\pi') > b(\pi)$$

Note that $b = s$ for both type π and π' immediately implies that we must have $a(\pi) > a(\pi')$, since

$$a(\pi) = c_2(\pi) - b(\pi) - T > c_2(\pi') - b(\pi') - T = a(\pi')$$

A similar argument can be used to prove $l(\pi)$ is decreasing in π.

Proof of 3) Restriction on prices (inequality (10)) implies that consumers of each type do not demand annuity and life insurance at the same time. For $\pi = 0$ there is no demand for annuity, $a(0) = 0$. If $a(1) = 0$ the claim is established, otherwise the existence of $\underline{\pi}_a$ is established by continuity (and monotonicity) of $a(\pi)$. The existence of $\bar{\pi}_l$ is also similar. We only need to prove that $\underline{\pi}_a > \bar{\pi}_l$. Let $\lambda^a(\pi)$ and $\lambda^l(\pi)$ be the multipliers on non-negativity constraints for $a(\pi)$ and $l(\pi)$ in consumer's problem. Then λ^a and λ^l are also continuous functions of π. $\lambda^a(\pi) = 0$ for all $\pi > \underline{\pi}_a$ and $\lambda^a(\pi) > 0$ for all $\pi < \underline{\pi}_a$. By continuity then we must have $\lambda^a(\underline{\pi}_a) = 0$. That is consumer of type $\underline{\pi}_a$ is indifferent between buying annuity or not. Similarly, it must be true that $\lambda^l(\bar{\pi}_l) = 0$. If, $\underline{\pi}_a = \bar{\pi}_l$, then $\lambda^a(\underline{\pi}_a) = \lambda^l(\bar{\pi}_l) = 0$. This means that first order conditions for a and l must hold with equality. But I argued in the proof of second part that this cannot happen when prices satisfy inequality (10). Therefore, we must have $\underline{\pi}_a > \bar{\pi}_l$ (note that $\underline{\pi}_a < \bar{\pi}_l$ is already ruled out by the fact that each consumer -at most- buys only one type of insurance).

Proof of Proposition 3

We need to prove that there are prices p^{a*} and p^{l*} that solve the equations (7) and (8). The strategy is to write these equations as a fixed point problem and use the *Brouwer's Fixed Point Theorem* to prove the existence of equilibrium prices. Define the following functions

$$h_a(p^a, p^l) = \begin{cases} \frac{\int_{\underline{\pi}_a(p^a,p^l)}^{\bar{\pi}} \pi a(\pi; p^a, p^l) d\mu(\pi)}{A \int_{\underline{\pi}_a(p^a,p^l)}^{\bar{\pi}} a(\pi; p^a, p^l) d\mu(\pi)} & \text{if } \underline{\pi}_a(p^a, p^l) < \bar{\pi} \\ \frac{\bar{\pi}}{A} & \text{if } \underline{\pi}_a(p^a, p^l) = \bar{\pi} \end{cases}$$

and

$$h_l(p^a, p^l) = \begin{cases} \frac{\int_{\underline{\pi}}^{\bar{\pi}_l(p^a,p^l)} (1-\pi) l(\pi; p^a, p^l) d\mu(\pi)}{A \int_{\underline{\pi}}^{\bar{\pi}_l(p^a,p^l)} l(\pi; p^a, p^l) d\mu(\pi)} & \text{if } \bar{\pi}_l(p^a, p^l) > \underline{\pi} \\ \frac{(1-\underline{\pi})}{A} & \text{if } \bar{\pi}_l(p^a, p^l) = \underline{\pi} \end{cases}$$

Where $\bar{\pi}_l(p^a, p^l)$ and $\underline{\pi}_a(p^a, p^l)$ are cut-off probabilities in consumer's problem at prices (p^a, p^l). First note that $\underline{\pi}_a(p^a, p^l)$ solves the following system of equations

$$u'(e(1-\tau) - q\widetilde{s}) \quad = \quad \widetilde{\pi} A\beta u'(Tr + \widetilde{s}) + (1 - \widetilde{\pi}) A\beta v'(\widetilde{s})$$

$$p^a u'(e(1-\tau) - q\widetilde{s}) \quad = \quad \widetilde{\pi}\beta u'(Tr + \widetilde{s})$$

Where the unknowns are $\widetilde{\pi}$ and $\widetilde{s}$, the cut-off probability and optimal holding of risk free security at the cut-off probability (for given prices)[8]. It is clear that $\underline{\pi}_a(p^a, p^l)$ is a continuous function of prices (direct application of implicit function theorem to the above equations). The same is true for $\bar{\pi}_l(p^a, p^l)$.

Now define the fixed point function

$$H(p^a, p^l) = (h_a(p^a, p^l), h_l(p^a, p^l))$$

And note that $\underline{\pi} \leq Ah_a(p^a, p^l) \leq \bar{\pi}$ and $1 - \bar{\pi} \leq Ah_l(p^a, p^l) \leq 1 - \underline{\pi}$.

The claim is that function $H(p^a, p^l)$ has a fixed point in $[\frac{1}{A}\underline{\pi}, \frac{1}{A}\bar{\pi}] \times [\frac{1}{A}(1-\bar{\pi}), \frac{1}{A}(1-\underline{\pi})]$. Aggregate demand for annuity and life insurance are continuous in prices for standard reasons. We only need to prove the continuity of $H(p^a, p^l)$ at those prices such that $\underline{\pi}_a(p^a, p^l) = \bar{\pi}$ or $\bar{\pi}_l(p^a, p^l) = \underline{\pi}$ (where aggregate demand is zero).

Suppose $\bar{p} = (\bar{p^a}, \bar{p^l})$ be a vector of price such that $\underline{\pi}_a(\bar{p}) = \bar{\pi}$. Take a sequence p_n converging to $\bar{p}$ and notice that

$$\bar{\pi} \geq Ah_a(p_n) \geq \underline{\pi}_a(p_n)$$

Now take the limit $p_n \to \bar{p}$ and by continuity of $\underline{\pi}_a(p)$ we get $\lim_{p_n \to \bar{p}} \underline{\pi}_a(p_n) = \bar{\pi}$ and therefore $A \lim_{p_n \to \bar{p}} h_a(p_n) = \bar{\pi}$ and $h_a(p_n)$ is continuous. The same is true for $h_l(p)$ and by *Brouwer's Fixed Point Theorem* there is a vector of prices $(p^{a*}, p^{l*}) \in [\frac{1}{A}\underline{\pi}, \frac{1}{A}\bar{\pi}] \times [\frac{1}{A}(1-\bar{\pi}), \frac{1}{A}(1-\underline{\pi})]$ such that $H(p^{a*}, p^{l*}) = (p^{a*}, p^{l*})$. This establishes the existence of equilibrium.

Proof of Theorem 1

Suppose that annuity market is active, i.e, $\int_{\underline{\pi}}^{\bar{\pi}} a(\pi) d\mu(\pi) \neq 0$ Then from the equation (7) we have

$$\bar{\pi} > Ap^{a*} > \underline{\pi}_a$$

[8]If $\tilde{s}$ turns out to be negative, we can set it equal zero and then $\tilde{\pi}$ is just the solution to the second equation with $\tilde{s} = 0$

and therefore, there is a type $\pi = Ap^{a*}$ that faces fair annuity price. All consumers of this type can achieve full insurance and their allocation satisfy

$$u'(c_1(\pi)) = \frac{\beta\pi}{p^{a*}} U'(c_2(\pi)) = \frac{\beta(1-\pi)}{(\frac{1}{A} - p^{a*})} v'(b(\pi))$$

together with the following budget constraint (which is derived by replacing for $a(\pi)$

$$c_1(\pi) + p^{a*}(c_2(\pi) - T) + (\frac{1}{A} - p^{a*})b(\pi) = e(1-\tau)$$

These equations are simplified to

$$u'(c_1(\pi)) = A\beta U'(c_2(\pi)) = A\beta v'(b(\pi))$$

$$c_1(\pi) + \frac{\pi}{A}(c_2(\pi) - T) + \frac{(1-\pi)}{A} b(\pi) = e(1-\tau)$$

and since the annuity purchase by this type is positive we have $l(\pi) = 0$ and $b(\pi) = s(\pi)$ and

$$c_2(\pi) - Tr - b(\pi) = a(\pi) > 0$$

Now suppose to the contrary that the market for life insurance is also active. Then

$$1 - \underline{\pi} > Ap^{l*} > 1 - \bar{\pi}_l$$

and there is a type $\tilde{\pi} = 1 - Ap^{l*} < \pi$ that faces fair life insurance price and chooses allocations that satisfy

$$u'(c_1(\tilde{\pi})) = A\beta U'(c_2(\tilde{\pi})) = A\beta v'(b(\tilde{\pi}))$$

$$c_1(\tilde{\pi}) + \frac{\tilde{\pi}}{A}(c_2(\tilde{\pi}) - T) + \frac{(1-\tilde{\pi})}{A} b(\tilde{\pi}) = e(1-\tau)$$

and since the consumers of this type purchases life insurance we have $a(\tilde{\pi}) = 0$ and $c_2(\tilde{\pi}) = T + As(\tilde{\pi})$. Also $b(\tilde{\pi}) = l(\tilde{\pi}) + As(\tilde{\pi})$. By subtracting these two we get

$$c_2(\tilde{\pi}) - T - b(\tilde{\pi}) = -l(\tilde{\pi}) < 0$$

But recall from the earlier lemma that when consumers are facing fair prices they choose either positive net annuity or negative (or zero). Here, consumer type π faces fair annuity prices and chooses positive (net) annuity purchase. At the same time consumer type $\tilde{\pi}$ faces fair life insurance prices and chooses positive life insurance (negative net annuity) purchase. As shown in the earlier lemma, this is a contradiction.

Proof of Theorem 2

Consider the cutoff τ^* constructed from the ex ante efficient allocation. The claim is that every consumer then chooses the efficient allocation and purchases no private insurance. By construction, these allocations satisfy every type's budget constraint since τ^* is such that

$$c_1^* + \frac{1}{A}b^* = (1-\tau^*)e$$

and

$$T^* = c_2^* - b^*$$

and $a^* = l^* = 0$ and $s^* = \frac{1}{A}b^*$, where starred allocations are efficient allocations. It remains to show that these allocations are optimal for all types. Suppose otherwise. Suppose a positive measure of types purchase annuities (and by the market-structure theorem no life insurance) and therefore the price of annuity is $\bar{\pi} > Ap^a > \underline{\pi}$. Then there must be a type $\pi = Ap^a$ that faces a fair price. This type chooses the starred allocation, since it satisfies that type's budget constraint and first order conditions. But this implies that all types with lower survival probability than $\pi = Ap^a$ would also choose zero annuity purchases. This in turn implies that the market price must be higher than $\frac{1}{A}\pi$, a contradiction. Therefore annuity purchase must be zero for all and $Ap^{a*} = \bar{\pi}$. Similarly it can be shown that there is no activity in life insurance market and $1 - Ap^{l*} = \underline{\pi}$.

For $\tau < \tau^*$, there is one type that faces fair prices (this type can be $\bar{\pi}$). As shown above, a consumer facing fair prices in this case chooses positive net annuity purchase. This means that this type will choose positive annuity (and zero life insurance). If this type's survival probability is $\bar{\pi}$, then by continuity of $a(\pi)$ there exists a neighborhood around $\bar{\pi}$ such that demand for annuity is positive. But this implies that the type that faces fair price must be lower than $\bar{\pi}$. If this type's survival probability is strictly less than $\bar{\pi}$, the claim is established and annuity market is active. Then argument can be used to show that for $\tau > \tau^*$ the life insurance market is active.

Proof of Proposition 4

The proof is immediate by construction of τ^*.

Proof of Proposition 5

Suppose $\tau < \tau^*$. Then we know there will be no demand for life insurance and $p^{l*} = \frac{1}{A} - \underline{\pi}$. Therefore $\frac{\partial p^{l*}}{\partial \tau} = 0$. Now consider the consumer problem (and we know $l(\pi) = 0$)

$$\max u(c_1) + \beta\pi U(c_2) + \beta(1-\pi)v(b)$$

subject to

$$\begin{aligned} c_1 + p^a a + s &\leq e(1-\tau) \\ c_2 &\leq T + a + As \\ b &\leq As \end{aligned}$$

replace for a and s in budget constraint and we get

$$c_1 + p^a c_2 + \left(\frac{1}{A} - p^a\right) b \leq (1-\tau)e + p^a T$$

Then because of homotheticity the solution will have the following form (if $\pi \geq \underline{\pi}_a$)

$$\begin{aligned} c_1 &= \phi_1(p^a, \pi)((1-\tau)e + p^a T) \\ c_2 &= \phi_2(p^a, \pi)((1-\tau)e + p^a T) \\ b &= \phi_b(p^a, \pi)((1-\tau)e + p^a T) \end{aligned}$$

where $\phi_1(p^a, \pi)$, $\phi_2(p^a, \pi)$ and $\phi_b(p^a, \pi)$ are between zero and one. Then we can find the demand for annuity

$$a(p^a, \pi, \tau) = \begin{cases} \phi_a(p^a, \pi)((1-\tau)e + p^a T) - T & \text{if } \pi \geq \underline{\pi}_a \\ 0 & \text{Otherwise} \end{cases}$$

where $\phi_a(p^a, \pi) = \phi_1(p^a, \pi) - \phi_b(p^a, \pi)$. I have already shown that annuity purchase is an increasing function of type, therefore

$$\frac{\partial \phi_a}{\partial \pi} > 0$$

also using $T = \frac{A\tau e}{\mathbf{E}[\pi]}$ we can show

$$\frac{\partial a}{\partial \tau} = \phi_a(p^a, \pi)(-e + \frac{Ap^a}{\mathbf{E}[\pi]}e) - \frac{Ae}{\mathbf{E}[\pi]}$$

$$= \quad -\phi_a(p^a,\pi)e - \frac{Ae}{\mathbf{E}[\pi]}(1 - p^a\phi_a(p^a,\pi)) < 0$$

Also we can determine the sign of the following cross derivative

$$\frac{\partial^2 a}{\partial\tau\partial\pi} = \frac{\partial\phi_a}{\partial\pi}\left(\frac{p^a Ae}{\mathbf{E}[\pi]} - e\right) > 0$$

where the last inequality follows from $Ap^a > \mathbf{E}[\pi]$, which was established before.
Next consider the price equation (7). Let's define the function $h(p^a,\tau)$ as

$$h(p^a,\tau) = \frac{\mathrm{E}_\pi[\pi a(p^a,\pi,\tau)|\pi \geq \underline{\pi}_a(p^a,\tau)]}{A\mathrm{E}_\pi[a(p^a,\pi,\tau)|\pi \geq \underline{\pi}_a(p^a,\tau)]}$$

Note that since we assume $\tau < \tau^*$ the aggregate demand for annuity is positive and the above expression is well defined. The goal is to show that $\frac{\partial h(p^{a*},\tau)}{\partial\tau} > 0$. But note first that

$$\begin{aligned}\frac{\partial\mathrm{E}_\pi[a(p^a,\pi,\tau)|\pi \geq \underline{\pi}_a(p^a,\tau)]}{\partial\tau} &= -\quad \frac{\partial\underline{\pi}_a(p^a,\tau)}{\partial\tau}a(p^a,\underline{\pi}_a(p^a,\tau),\tau) + \mathrm{E}_\pi[\frac{\partial a}{\partial\tau}|\pi \geq \underline{\pi}_a(p^a,\tau)] \\ &= \quad \mathrm{E}_\pi[\frac{\partial a}{\partial\tau}|\pi \geq \underline{\pi}_a(p^a,\tau)]\end{aligned}$$

where the last equality is true because by definition $a(p^a,\underline{\pi}_a(p^a,\tau),\tau) = 0$. Similarly we have

$$\frac{\partial\mathrm{E}_\pi[\pi a(p^a,\pi,\tau)|\pi \geq \underline{\pi}_a(p^a,\tau)]}{\partial\tau} = \mathrm{E}_\pi[\pi\frac{\partial a}{\partial\tau}|\pi \geq \underline{\pi}_a(p^a,\tau)]$$

Now we can determine the sign of the derivative

$$\begin{aligned}\frac{\partial h(p^{a*},\tau)}{\partial\tau} &= \frac{\mathrm{E}_{\pi\geq\underline{\pi}_a(p^a,\tau)}[\pi\frac{\partial a}{\partial\tau}]\mathrm{E}_{\pi\geq\underline{\pi}_a(p^a,\tau)}[a(p^a,\pi,\tau)] - \mathrm{E}_{\pi\geq\underline{\pi}_a(p^a,\tau)}[\frac{\partial a}{\partial\tau}]\mathrm{E}_{\pi\geq\underline{\pi}_a(p^a,\tau)}[\pi a(p^a,\pi,\tau)]}{A(\mathrm{E}_\pi[a(p^a,\pi,\tau)|\pi \geq \underline{\pi}_a(p^a,\tau)])^2} \\ &= \frac{\mathrm{E}_{\pi\geq\underline{\pi}_a(p^a,\tau)}[\pi\frac{\partial a}{\partial\tau}] - \mathrm{E}_{\pi\geq\underline{\pi}_a(p^a,\tau)}[\frac{\partial a}{\partial\tau}]Ah(p^{a*},\tau)}{A\mathrm{E}_\pi[a(p^a,\pi,\tau)|\pi \geq \underline{\pi}_a(p^a,\tau)]} \\ &= \frac{\mathrm{E}_{\pi\geq\underline{\pi}_a(p^a,\tau)}[(\pi - Ah(p^{a*},\tau))\frac{\partial a}{\partial\tau}]}{A\mathrm{E}_\pi[a(p^a,\pi,\tau)|\pi \geq \underline{\pi}_a(p^a,\tau)]} \\ &= \frac{\mathrm{E}_{\pi\geq\underline{\pi}_a(p^a,\tau)}[\frac{\partial a}{\partial\tau}]\,\mathrm{E}_{\pi\geq\underline{\pi}_a(p^a,\tau)}[\pi - Ah(p^{a*},\tau)]}{A\mathrm{E}_\pi[a(p^a,\pi,\tau)|\pi \geq \underline{\pi}_a(p^a,\tau)]} \\ &\quad + \frac{\mathrm{Cov}_{\pi\geq\underline{\pi}_a(p^a,\tau)}[(\pi - Ah(p^{a*},\tau))\frac{\partial a}{\partial\tau}]}{A\mathrm{E}_\pi[a(p^a,\pi,\tau)|\pi \geq \underline{\pi}_a(p^a,\tau)]} > 0\end{aligned}$$

The last inequality is true because the first term in the numerator is the product of two negative terms ($\frac{\partial a}{\partial\tau} < 0$ is shown above and $\mathrm{E}_{\pi\geq\underline{\pi}_a(p^a,\tau)}[\pi - Ah(p^{a*},\tau)] < 0$ follows from the earlier lemma and was established in the proof of the market-structure theorem). The covariance term is positive because $(\pi - Ah(p^{a*}, \tau))$ is positively correlated with π and since $\frac{\partial^2 a}{\partial \tau \partial \pi} > 0$, the two terms are positively correlated.

Therefore, $h(p^{a*}, \tau)$ is an increasing function of τ at equilibrium price. Therefore, the equilibrium must increase with τ.